

CUNTZ-PIMSNER ALGEBRAS AND TWISTED TENSOR PRODUCTS

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ABSTRACT. Given two correspondences X and Y and a discrete group G which acts on X and coacts on Y , one can define a twisted tensor product $X \boxtimes Y$ which simultaneously generalizes ordinary tensor products and crossed products by group actions and coactions. We show that, under suitable conditions, the Cuntz-Pimsner algebra of this product, $\mathcal{O}_{X \boxtimes Y}$, is isomorphic to a “balanced” twisted tensor product $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$ of the Cuntz-Pimsner algebras of the original correspondences. We interpret this result in several contexts and connect it to existing results on Cuntz-Pimsner algebras of crossed products and tensor products.

1. INTRODUCTION

In the author’s previous paper [Mor15], it was shown that, under suitable conditions, if X and Y are C^* -correspondences over C^* -algebras A and B , the Cuntz-Pimsner algebra $\mathcal{O}_{X \otimes Y}$ is isomorphic to a subalgebra $\mathcal{O}_X \otimes_{\mathbb{T}} \mathcal{O}_Y$ of $\mathcal{O}_X \otimes \mathcal{O}_Y$. In the present paper, we will extend this result from ordinary tensor products to a certain class of “twisted” tensor products.

Many constructions in operator algebras may be thought of as “twisted” tensor products, for example: crossed products by actions or coactions of groups, \mathbb{Z}_2 -graded tensor products and so on. In [MRW14], a very general construction of a “twisted tensor product” is presented. Their construction involves two quantum groups $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$, two coactions $(A, \mathbb{G}, \delta_A)$ and $(B, \mathbb{H}, \delta_B)$, and a bicharacter $\chi \in \mathcal{U}(\widehat{S} \otimes \widehat{T})$. Given this information, they define a twisted tensor product $A \boxtimes_{\chi} B$. They also show that if X and Y are correspondences over A and B with compatible coactions of \mathbb{G} and \mathbb{H} , there is a natural way of defining a correspondence $X \boxtimes_{\chi} Y$ over $A \boxtimes_{\chi} B$. In this paper, we will work with a special case of this general construction which is general enough to be useful but simple enough to be very tractable.

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Specifically, we are interested in the case where, for some discrete group G , $S = c_0(G)$, $T = C_r^*(G)$ and $\chi = W^G \in \mathcal{U}(\widehat{C_r^*(G)} \otimes C_r^*(G))$ is the reduced bicharacter of $C_r^*(G)$ viewed as a quantum group. In this case, we may view the coaction of $c_0(G)$ as an action of G on A (or X), and we will be able to describe most of the algebraic properties of $A \boxtimes_\chi B$ and $X \boxtimes_\chi Y$ entirely in terms of elementary tensors.

In this simplified setting, we will prove our main result: if $J_{X \boxtimes_\chi Y} = J_X \boxtimes_\chi J_Y$ then $\mathcal{O}_{X \boxtimes_\chi Y} \cong \mathcal{O}_X \boxtimes_\chi \mathcal{O}_Y$ (where $J_X = \phi^{-1}(\mathcal{K}(X)) \cap (\ker(\phi))^\perp$ is the Katsura ideal). We will then apply this result to some specific examples.

2. PRELIMINARIES

2.1. Correspondences and Cuntz-Pimsner Algebras. For a general reference on correspondences, we refer the reader to [Lan95]. For Cuntz-Pimsner algebras, we recommend [Kat04], [Kat03] and the brief overview in [Rae05]. We will briefly recall some of the basic facts here.

Suppose A is a C^* -algebra and X is a right A -module. We say that X has an A -valued inner product if there is a map

$$X \times X \ni (x, y) \mapsto \langle x, y \rangle_A \in A$$

which is A -linear in the second variable and satisfies the following

- (1) $\langle x, x \rangle_A \geq 0$ for all $x \in X$ with equality if and only if $x = 0$
- (2) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ for all $x, y \in X$
- (3) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$ for all $x, y \in X$ and $a \in A$.

We can define the following norm on X :

$$\|x\|_A := \|\langle x, x \rangle_A\|_A^{\frac{1}{2}}$$

If X is complete under the norm $\|\cdot\|_A$ defined above, X is called a *right Hilbert A -module*. Note that if $A = \mathbb{C}$ then X is just a Hilbert space and we can think of Hilbert modules as generalized Hilbert spaces where the scalars are elements of some C^* -algebra A . Sometimes we will write (X, A) or X_A if we wish to emphasize A .

Let A be a C^* -algebra and let X be a right Hilbert A -module. Suppose $T : X \rightarrow X$ is an A -module homomorphism. If there is an A -module homomorphism T^* such that

$$\langle T^*x, y \rangle_A = \langle x, Ty \rangle_A$$

for all $x, y \in X$, then we call T *adjointable* and we refer to T^* as the *adjoint* of T . The set of all adjointable operators on X with the operator norm is a C^* -algebra. We denote this algebra by $\mathcal{L}(X)$. Given $x, y \in X$ we define the operator $\Theta_{x,y}$ as follows: $\Theta_{x,y}(z) = x\langle y, z \rangle_A$.

The closed linear span of all such operators is a subalgebra of $\mathcal{L}(X)$ which we call the *generalized compact operators*. It is denoted by $\mathcal{K}(X)$.

Suppose X is a right Hilbert B -module. Suppose further that we have a homomorphism $\phi : A \rightarrow \mathcal{L}(X)$ for some C^* -algebra A . This is called a *left action of A by adjointable operators*. We call the triple (A, X, B) a C^* -*correspondence* or simply a *correspondence*. For $a \in A$ and $x \in X$, we will write $a \cdot x$ for $\phi(a)(x)$. If $A = B$ we call this a *correspondence over A (or B)*. We call the left-action *injective* if ϕ is injective and *non-degenerate* if $\phi(A)X = X$. If $\phi(A) \subseteq \mathcal{K}(X)$, we say that the left action is *implemented by compacts*. We will sometimes write ${}_AX_B$ to indicate that X is an $A - B$ correspondence. Given an $A_1 - B_2$ correspondence X and an $A_2 - B_2$ correspondence Y , a *correspondence isomorphism* is a triple $(\varphi_A, \Phi, \varphi_B)$ where Φ is a linear isomorphism $\Phi : X \rightarrow Y$ and $\varphi_A : A_1 \rightarrow A_2$ and $\varphi_B : B_1 \rightarrow B_2$ are isomorphisms of C^* -algebras such that the left and right actions and the inner product are preserved by the maps:

$$\begin{aligned}\Phi(ax) &= \varphi_A(a)\Phi(x) \\ \Phi(xb) &= \Phi(x)\varphi_B(b) \\ \langle \Phi(x), \Phi(x') \rangle_B^Y &= \varphi_B(\langle x, x' \rangle_B^X)\end{aligned}$$

Where $\langle \cdot, \cdot \rangle_B^Y$ denotes the B -valued inner product on Y and $\langle \cdot, \cdot \rangle_B^X$ denotes the B -valued inner product on X . It will be convenient to introduce the following definition:

Definition 2.1. Let X be an $A - B$ correspondence. A *generating system* for X is a triple (A^0, X^0, B^0) where $A^0 \subseteq A$, $X^0 \subseteq X$ and $B^0 \subseteq B$ such that $\overline{\text{span}}(A^0) = A$, $\overline{\text{span}}(X^0) = X$, and $\overline{\text{span}}(B^0) = B$ and such that for all $x \in X^0$ we have that $ax, xb \in X^0$ for all $a \in A^0$ and $b \in B^0$. If $A = B$ and $A^0 = B^0$ we will denote the generating system by (X^0, A^0) .

We will make frequent use of the following fact:

Lemma 2.2. Let X be an $A_1 - B_1$ correspondence and Y be a $A_2 - B_2$ correspondence. Suppose that (A_1^0, X^0, B_1^0) and (A_2^0, Y^0, B_2^0) are generating sets for X and Y respectively. Let $\varphi_A : A_1 \rightarrow A_2$ and $\varphi_B : B_1 \rightarrow B_2$ be isomorphisms. Suppose there is a bijection $\Phi_0 : X^0 \rightarrow Y^0$, which preserves the inner product, left and right actions, and scalar

multiplication. That is

$$\begin{aligned}
\langle \Phi_0(x), \Phi_0(x') \rangle_B^Y &= \varphi_B \left(\langle x, x' \rangle_B^X \right) && \text{for all } x, x' \in X^0 \\
\Phi_0(ax) &= \varphi_A(a) \Phi_0(x) && \text{for all } x \in X^0 \text{ and } a \in A^0 \\
\Phi_0(xb) &= \Phi_0(x) \varphi_B(b) && \text{for all } x \in X^0 \text{ and } b \in B^0 \\
\Phi_0(cx) &= c \Phi_0(x) && \text{for all } x \in X^0 \text{ and } c \in \mathbb{C}
\end{aligned}$$

Then Φ_0 extends linearly and continuously to a correspondence isomorphism $\Phi : X \rightarrow Y$.

Proof. Let $x \in \text{span}(X^0)$, then $x = \sum_i c_i x_i$ for some $x_i \in X^0$ and $c_i \in \mathbb{C}$. We define $\Phi(x) = \sum_i c_i \Phi_0(x_i)$. First, we must verify that Φ is well defined on $\text{span}(X^0)$. Suppose $\sum_{i=1}^n c_i x_i$ and $\sum_{i=1}^m d_i x'_i$ are both equal to $x \in X$ with $x_i, x'_i \in X^0$. Let $y = \sum_{i=1}^n c_i \Phi_0(x_i)$ and $y' = \sum_{i=1}^m d_i \Phi_0(x'_i)$. Then

$$\begin{aligned}
\|y - y'\|^2 &= \langle y - y', y - y' \rangle_{B'} \\
&= \left\langle \sum_{i=1}^n c_i \Phi_0(x_i) - \sum_{i=1}^m d_i \Phi_0(x'_i), \sum_{i=1}^n c_i \Phi_0(x_i) - \sum_{i=1}^m d_i \Phi_0(x'_i) \right\rangle_{B'} \\
&= \sum_{i,j=1}^{n,n} \bar{c}_i c_j \langle \Phi_0(x_i), \Phi_0(x_j) \rangle_{B'} + \sum_{i,j=1}^{m,m} \bar{d}_i d_j \langle \Phi_0(x'_i), \Phi_0(x'_j) \rangle_{B'} \\
&\quad - \sum_{i,j=1}^{n,m} \bar{c}_i d_j \langle \Phi_0(x_i), \Phi_0(x'_j) \rangle_{B'} - \sum_{i,j=1}^{m,n} \bar{d}_i c_j \langle \Phi_0(x'_i), \Phi_0(x_j) \rangle_{B'} \\
&= \sum_{i,j=1}^{n,n} \bar{c}_i c_j \varphi_B(\langle x_i, x_j \rangle_B) + \sum_{i,j=1}^{m,m} \bar{d}_i d_j \varphi_B(\langle x'_i, x'_j \rangle_B) \\
&\quad - \sum_{i,j=1}^{n,m} \bar{c}_i d_j \varphi_B(\langle x_i, x'_j \rangle_B) - \sum_{i,j=1}^{m,n} \bar{d}_i c_j \varphi_B(\langle x'_i, x_j \rangle_B) \\
&= \varphi_B \left(\left\langle \sum_{i=1}^n c_i x_i - \sum_{i=1}^m d_i x'_i, \sum_{i=1}^n c_i x_i - \sum_{i=1}^m d_i x'_i \right\rangle_B \right) \\
&= \varphi_B(\langle x - x, x - x \rangle_B) \\
&= 0
\end{aligned}$$

where we have used the fact that φ is an isomorphism and thus linear.

For $x, x' \in X^0$ we have

$$\begin{aligned}
\langle \Phi(x), \Phi(x') \rangle_B^Y &= \left\langle \sum_i c_i \Phi_0(x_i), \sum_j c'_j \Phi_0(x'_j) \right\rangle_B^Y \\
&= \sum_{i,j} \bar{c}_i c'_j \langle \Phi_0(x_i), \Phi_0(x'_j) \rangle_B^Y \\
&= \sum_{i,j} \bar{c}_i c'_j \varphi_B \left(\langle x_i, x'_j \rangle_B^X \right) \\
&= \varphi_B \left(\left\langle \sum_i c_i x_i, \sum_j c'_j x'_j \right\rangle_B^X \right) \\
&= \varphi_B \left(\langle x, x' \rangle_B^X \right)
\end{aligned}$$

Therefore, Φ preserves the inner product on $\text{span}(X^0)$ and thus also preserves the norm on $\text{span}(X^0)$ and so Φ is bounded and can be extended continuously to $\overline{\text{span}}(X^0) = X$. Hence, for any $z \in X$ we can approximate $z \approx \sum_i c_i z_i$ with $z_i \in X^0$. Thus for any $a_0 \in A^0$ we have

$$\Phi(a_0 z) \approx \Phi\left(a_0 \sum_i c_i z_i\right) = \sum_i c_i \Phi(a_0 z_i) = \varphi_A(a_0) \sum_i c_i \Phi(z_i) \approx \varphi_A(a_0) \Phi(z)$$

so $\Phi(a_0 z) = \varphi_A(a_0) \Phi(z)$ and similarly $\Phi(z b_0) = \Phi(z) \varphi_B(b_0)$ for $b_0 \in B^0$. For arbitrary $a \in A$ we may approximate $a \approx \sum_i c_i a_i$ with $a_i \in A^0$ and we see that

$$\Phi(a z) \approx \sum_i c_i \Phi(a_i z) = \varphi_A\left(\sum_i c_i a_i\right) \Phi(z) \approx \varphi_A(a) \Phi(z)$$

So $\Phi(a z) = \varphi_A(a) \Phi(z)$ for all $a \in A$ and similarly $\Phi(z b) = \Phi(z) \varphi_B(b)$ for all $b \in B$. We already know that Φ is injective since it preserves the norm, so we only have to show that it is surjective. To see this note that

$$\Phi(X) = \overline{\text{span}}(\Phi_0(X^0)) = \overline{\text{span}}(Y^0) = Y$$

So we have established that Φ is a linear isomorphism from X to Y which preserves the left and right actions and the inner product, in other words Φ is a correspondence isomorphism. \square

Given a Hilbert module (X, A) , we define the *linking algebra* of (X, A) to be the C^* -algebra $L(X) := \mathcal{K}(X \oplus A)$. There are complimentary projections p and q in $M(L(X))$ such that $pL(X)p \cong \mathcal{K}(X)$, $pL(X)q \cong X$, and $qL(X)q \cong A$. This gives $L(X)$ the following block matrix decomposition:

$$L(X) = \begin{bmatrix} \mathcal{K}(X) & X \\ \overline{X} & A \end{bmatrix}$$

with $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. The benefit of using linking algebras is that the algebraic properties of X are encoded into the multiplicative structure of $L(X)$:

$$\begin{aligned} (1) \quad & \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} 0 & xa \\ 0 & 0 \end{bmatrix} \\ (2) \quad & \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}^* \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \langle x, y \rangle_A \end{bmatrix} \\ (3) \quad & \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} \Theta_{x,y} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Given a correspondence X over A , a *Toeplitz representation* of (X, A) in a C^* -algebra B is a pair (ψ, π) where $\psi : X \rightarrow B$ is a linear map and $\pi : A \rightarrow B$ is a homomorphism satisfying:

$$\begin{aligned} \psi(xa) &= \psi(x)\pi(a) \\ \pi(\langle x, y \rangle_A) &= \psi(x)^*\psi(y) \\ \psi(ax) &= \pi(a)\psi(x) \end{aligned}$$

By $C^*(\psi, \pi)$ we shall mean the C^* -subalgebra of B generated by the images of ψ and π in B . There is a unique (up to isomorphism) C^* -algebra \mathcal{T}_X , called the *Toeplitz algebra* of X , which is generated by a universal Toeplitz representation (i_X, i_A) .

Let $X^{\otimes n}$ denote the n -fold internal tensor product (see [Lan95] for information on internal tensor products) of X with itself. By convention, we let $X^{\otimes 0} = A$. Let (ψ, π) be a Toeplitz representation of X in B . Define the map $\psi^n : X^{\otimes n} \rightarrow B$ for each $n \in \mathbb{N}$ as follows: let $\psi^0 = \pi$, $\psi^1 = \psi$, and set $\psi^n(x \otimes y) = \psi(x)\psi^{n-1}(y)$ (where $x \in X$ and $y \in X^{\otimes n-1}$) for each $n > 1$.

We summarize Proposition 2.7 of [Kat04] as follows: Let (ψ, π) be a Toeplitz representation of a correspondence X over A . Then

$$C^*(\psi, \pi) = \overline{\text{span}}\{\psi^n(x)\psi^m(y)^* : x \in X^{\otimes n}, y \in X^{\otimes m}\}$$

By Lemma 2.4 of [Kat04], for each $n \in \mathbb{N}$ there is a homomorphism $\psi^{(n)} : \mathcal{K}(X^{\otimes n}) \rightarrow B$ such that:

- (1) $\pi(a)\psi^{(n)}(k) = \psi^{(n)}(\phi(a)k)$ for all $a \in A$ and all $k \in \mathcal{K}(X^{\otimes n})$.
- (2) $\psi^{(n)}(k)\psi(x) = \psi(kx)$ for all $x \in X$ and all $k \in \mathcal{K}(X^{\otimes n})$.

We define the *Katsura ideal* of A to be the ideal:

$$J_X = \{a \in A : \phi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker(\phi)\}$$

Where ϕ is the left action. This is often written as $J_X = \phi^{-1}(\mathcal{K}(X)) \cap (\ker(\phi))^\perp$. In many cases of interest, the left action of a correspondence is injective and implemented by compact operators. In this case we have $J_X = A$.

A Toeplitz representation is said to be *Cuntz-Pimsner covariant* if $\psi^{(1)}(\phi(a)) = \pi(a)$ for all $a \in J_X$. The *Cuntz-Pimsner algebra* \mathcal{O}_X is the quotient of \mathcal{T}_X by the ideal generated by

$$\{i_X^{(1)}(\phi(a)) - i_A(a) : a \in J_X\}$$

It can be shown that \mathcal{O}_X is generated by a universal Cuntz-Pimsner covariant representation (k_X, k_A) .

2.2. Actions and Coactions of Discrete Groups. Working with actions and coactions of locally compact groups and their crossed products can be somewhat complicated, for a general reference we suggest the appendix of [EKQR02]. Restricting attention to discrete groups affords many simplifications. In this section, we will summarize some of these simplifications.

Proposition 2.3. *Let G be a discrete group and let α be an action of G on a C^* -algebra A . Let (i_A, i_G) be the canonical representation of the system (A, G, α) in the reduced crossed product $A \rtimes_{\alpha, r} G$. Then $A \rtimes_{\alpha, r} G$ is the closed linear span of elements of the form $i_A(a)i_G(s)$ where $a \in A$ and $s \in G$. These have the following algebraic properties:*

$$(4) \quad (i_A(a)i_G(s))(i_A(b)i_G(t)) = i_A(a\alpha_s(b))i_G(st)$$

$$(5) \quad (i_A(a)i_G(s))^* = i_A(\alpha_{s^{-1}}(a^*))i_G(s^{-1})$$

Proof. The fact that the $i_A(a)i_G(s)$ densely span $A \rtimes_{\alpha, r} G$ follows from the definition of the crossed product. The algebraic properties follow easily from the fact that $i_A(a)i_G(s) = i_G(s)i_A(\alpha_s(a))$ which we obtain from $i_G(s)^*i_A(a)i_G(s) = i_A(\alpha_s(a))$. \square

Proposition 2.4. *Let (γ, α) be an action of a discrete group G on a correspondence (X, A) . Let (i_X, i_A, i_G^X, i_G^A) be the canonical representation of the system in the crossed product $(X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G)$. Then $X \rtimes_{\gamma, r} G$ is the closed linear span of $i_X(x)i_G^X(s)$ where $x \in X$ and $s \in G$. These satisfy the following algebraic properties:*

$$(6) \quad (i_X(x)i_G^X(s))(i_A(a)i_G^A(t)) = i_X(x\alpha_s(a))i_G^X(st)$$

$$(7) \quad \langle i_X(x)i_G^X(s), i_X(y)i_G^Y(t) \rangle_{A \rtimes_{\alpha, r} G} = i_A(\alpha_{s^{-1}}(\langle x, y \rangle_A))i_G^A(s^{-1}t)$$

$$(8) \quad (i_A(a)i_G^A(s))(i_X(x)i_G^X(t)) = i_X(a\gamma_s(x))i_G^X(st)$$

Proof. The fact that the $i_X(x)i_G^X(s)$ densely span $X \rtimes_{\gamma,r} G$ follows from the definition of the crossed product. To understand the algebraic properties, recall from Lemma 3.3 of [EKQR02] that $L(X \rtimes_{\gamma,r} G) \cong L(X) \rtimes_{\nu,r} G$ where ν is the coaction on $L(X)$ induced by (γ, α) . (6) and (7) are then easily deduced by applying the previous proposition to $L(X) \rtimes_{\nu,r} G$. (8) follows from the fact that the left action must be covariant with respect the action. \square

Corollary 2.5. *The sets*

$$\begin{aligned} (X \rtimes_{\gamma,r} G)_0 &:= \{i_X(x)i_G^X(s) : x \in X, s \in G\} \\ (A \rtimes_{\alpha,r} G)_0 &:= \{i_A(a)i_G^A(s) : a \in A, s \in G\} \end{aligned}$$

form a generating system for $X \rtimes_{\gamma,r} G$ in the sense of Definition 2.1.

The simplification of crossed products by coactions of discrete groups come from the realization that coactions by discrete groups can be viewed as gradings. This idea is presented in detail in [Qui96], but we briefly summarize the main points in the next two propositions. We refer the reader to [Qui96] for the proofs.

Proposition 2.6. *Let $\delta : A \rightarrow M(A \otimes C^*(G))$ be a coaction of a discrete group G on a C^* -algebra A . Then $A = \overline{\text{span}}\{A_s\}_{s \in G}$ where $A_s = \{a \in A : \delta(a) = a \otimes u_s\}$. Furthermore,*

$$(9) \quad A_s \cdot A_t \subseteq A_{st}$$

$$(10) \quad A_s^* = A_{s^{-1}}$$

Proposition 2.7. *Let δ be a coaction of a discrete group G on a C^* -algebra A and let (j_A, j_G) be the canonical representations of A and $c_0(G)$ in the crossed product $A \rtimes_\delta G$. Then $A \rtimes_\delta G$ is densely spanned by elements of the form $j_A(a_s)j_G(f)$ where $a_s \in A_s$, $f \in c_0(G)$ and These satisfy the following relations:*

$$(11) \quad (j_A(a)j_G(f))(j_A(a_s)j_G(g)) = j_A(aa_s)j_G(\lambda_{s^{-1}}(f)g)$$

$$(12) \quad (j_A(a_s)j_G(f))^* = j_A(a_s^*)j_G(\overline{f})$$

where λ_s denotes left translation by s on $c_0(G)$.

Applying these propositions to linking algebras helps us to understand coactions on correspondences:

Proposition 2.8. *Let (σ, δ) be a coaction of a discrete group G on a Hilbert module (X, A) . Then $X = \overline{\text{span}}\{X_s\}_{s \in G}$ where $X_s = \{x \in X :$*

$\sigma(x) = x \otimes u_s\}$. Further:

$$(13) \quad X_s \cdot A_t \subseteq X_{st}$$

$$(14) \quad \langle x_s, x_t \rangle_A \in A_{s^{-1}t}$$

$$(15) \quad A_s \cdot X_t \subseteq X_{st}$$

Proof. Let ε be the induced coaction on $L(X)$. Then we have a grading $L(X) = \overline{\text{span}}\{L(X)_s\}_{s \in G}$. Since p and q are in fixed points of the coaction, if $z \in L(X)_s$ then $qzq \in L(X)_s$ and $pzq \in L(X)_s$. Recall that the restriction of ε to $qL(X)q = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \cong A$ is δ . Thus if a is the element of A corresponding to qzq then $\varepsilon(qzq) = (qzq) \otimes u_s$ if and only if $\delta(a) = a \otimes u_s$. Thus $qL(X)_sq = \begin{bmatrix} 0 & 0 \\ 0 & A_s \end{bmatrix} \cong A_s$. Similar reasoning shows that $pL(X)_sq = \begin{bmatrix} 0 & X_s \\ 0 & 0 \end{bmatrix} \cong X_s$. (13) and (14) then follow from multiplication in $L(X)$ together with the grading of $L(X)$. (15) follows from the fact that the left module action is covariant with respect to the coaction. \square

Lemma 3.4 of [EKQR02] shows that $L(X \rtimes_\sigma G) \cong L(X) \rtimes_\varepsilon G$. This fact, together with the preceding propositions, gives us the following characterization of $X \rtimes_\sigma G$ in the case where G is discrete:

Proposition 2.9. *Let (σ, δ) be a coaction of a discrete group G on a correspondence (X, A) . Let (j_X, j_A, j_G^X, j_G^A) be the canonical representation of the system in the correspondence $(X \rtimes_\sigma G, A \rtimes_\delta G)$. Then $X \rtimes_\sigma G$ is densely spanned by elements of the form $j_X(x_s)j_G^X(f)$ where $x_s \in X_s$ and $f \in c_0(G)$. These elements satisfy the following relations:*

$$(16) \quad (j_X(x)j_G^X(f))(j_A(a_s)j_G^A(g)) = j_X(xa_s)j_G^X(\lambda_{s^{-1}}(f)g)$$

$$(17) \quad \langle j_X(x_s)j_G^X(f), j_X(x_t)j_G^X(g) \rangle_{A \rtimes_\delta G} = j_A(\langle x_s, x_t \rangle_A)j_G^A(\lambda_{t^{-1}s}(f)g)$$

$$(18) \quad (j_A(a)j_G^A(f))(j_X(x_s)j_G^X(g)) = j_X(ax_s)j_G^X(\lambda_{s^{-1}}(f)g)$$

Corollary 2.10. *The sets*

$$(X \rtimes_\sigma G)_0 := \{j_X(x_s)j_G^X(f) : x_s \in X_s, f \in c_0(G)\}$$

$$(A \rtimes_\delta G)_0 := \{j_A(a_s)j_G^A(f) : a_s \in A_s, f \in c_0(G)\}$$

form a generating system for $X \rtimes_\sigma G$.

2.3. Quantum Groups. There are many different notions of a quantum group. Our quantum groups will be quantum groups generated by modular multiplicative unitaries. We will briefly recall some of the

basic facts about these quantum groups and refer the reader to [Tim08] for a more in depth overview of the subject.

Definition 2.11. Given a separable Hilbert space \mathcal{H} , a *multiplicative unitary* \mathbb{W} is a unitary operator on $\mathcal{H} \otimes \mathcal{H}$ such that

$$(19) \quad \mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$$

where \mathbb{W}_{ij} indicates that we are applying \mathbb{W} to the i^{th} and j^{th} factors of \mathcal{H} and leaving the other fixed. Equation 19 is sometimes referred to as the *pentagon equation*.

\mathbb{W} is called *modular* if there exist (possibly unbounded) operators \widehat{Q} and Q on \mathcal{H} and a unitary $\widetilde{\mathbb{W}} \in \mathcal{U}(\overline{\mathcal{H}} \otimes \mathcal{H})$ (where $\overline{\mathcal{H}}$ is the dual space of \mathcal{H}) such that

- (1) \widehat{Q} and Q are positive and self-adjoint with trivial kernels
- (2) $\mathbb{W}^*(\widehat{Q} \otimes Q)\mathbb{W} = \widehat{Q} \otimes Q$
- (3) $\langle \eta' \otimes \xi', \mathbb{W}(\eta \otimes \xi) \rangle = \langle \overline{\eta} \otimes Q\xi', \widetilde{\mathbb{W}}(\overline{\eta'} \otimes Q^{-1}\xi) \rangle$ for all $\xi \in \text{Dom}(Q^{-1})$, $\xi' \in \text{Dom}(Q)$, and $\eta, \eta' \in \mathcal{H}$

Example 2.12 (Example 7.1.4 of [Tim08]). Let G be a locally compact group. We can identify $L^2(G) \otimes L^2(G)$ with $L^2(G \times G)$ and define $\mathbb{W}_G \in \mathcal{B}(L^2(G) \otimes L^2(G))$ by

$$(\mathbb{W}_G \zeta)(s, t) := \zeta(s, s^{-1}t)$$

Then \mathbb{W}_G is a multiplicative unitary.

Theorem 2.13 (Theorem 2.7 of [MRW14]). *For a modular multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, set*

$$S := \overline{\text{span}}\{(\omega \otimes \text{id}_{\mathcal{H}})\mathbb{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}$$

$$\widehat{S} := \overline{\text{span}}\{(\text{id}_{\mathcal{H}} \otimes \omega)\mathbb{W} : \omega \in \mathcal{B}(\mathcal{H})_*\}$$

Then:

- S and \widehat{S} are separable, nondegenerate C^* -subalgebras of $\mathcal{B}(\mathcal{H})$.
- $\mathbb{W} \in \mathcal{U}(\widehat{S} \otimes S) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$. When we wish to view \mathbb{W} as a unitary multiplier of $\widehat{S} \otimes S$ we will denote it by W^S and refer to it as the reduced bicharacter of S .
- There are unique homomorphisms $\Delta_S, \widehat{\Delta}_S : S \rightarrow S \otimes S$ such that

$$(\text{id}_{\widehat{S}} \otimes \Delta_S)W^S = W_{12}^S W_{13}^S \in \mathcal{U}(\widehat{S} \otimes S \otimes S)$$

$$(\widehat{\Delta}_S \otimes \text{id}_S)W^S = W_{23}^S W_{13}^S \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes S)$$

- (S, Δ_S) and $(\widehat{S}, \widehat{\Delta}_S)$ are C^* -bialgebras

Definition 2.14. A *quantum group* is a C^* -bialgebra $\mathbb{G} := (S, \Delta_S)$ arising from a modular multiplicative unitary as in Theorem 2.13. The *dual* $\widehat{\mathbb{G}}$ of \mathbb{G} is the bialgebra $(\widehat{S}, \widehat{\Delta}_S)$. $\widehat{\mathbb{G}}$ is generated by the modular multiplicative unitary $\widehat{\mathbb{W}} := \Sigma \mathbb{W}^* \Sigma$ (where Σ is the flip isomorphism: $x \otimes y \mapsto y \otimes x$). $\widehat{\mathbb{W}}$ is called the *dual* of \mathbb{W} . We will sometimes write $S(\mathbb{W})$ and $\widehat{S}(\mathbb{W})$ for S and \widehat{S} if we wish to call attention to the multiplicative unitary which generated S or \widehat{S} .

Example 2.15 (Example 7.2.13 of [Tim08]). Let G be a locally compact group and let \mathbb{W}_G be the multiplicative unitary described in Example 2.12. Then $S(\mathbb{W}_G) \cong C_r^*(G)$ and $\widehat{S}(\mathbb{W}_G) \cong C_0(G)$ as C^* -bialgebras. We denote the reduced bicharacter associated to \mathbb{W}_G by W_G .

Definition 2.16. Given two quantum groups $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$, we define a *bicharacter from \mathbb{G} to \mathbb{H}* to be a unitary $\chi \in \mathcal{U}(\widehat{S} \otimes \widehat{T})$ such that

$$(20) \quad (\widehat{\Delta}_S \otimes \text{id}_{\widehat{T}})\chi = \chi_{23}\chi_{13} \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes \widehat{T})$$

$$(21) \quad (\text{id}_{\widehat{S}} \otimes \widehat{\Delta}_T)\chi = \chi_{13}\chi_{12} \in \mathcal{U}(\widehat{S} \otimes \widehat{S} \otimes \widehat{T})$$

Bicharacters are used in the construction of the twisted tensor product. The reduced bicharacter from Theorem 2.13 is a special case of a bicharacter. In fact the only bicharacter we will need for our simplified twisted tensor products is the bicharacter W_G associated to the multiplicative unitary \mathbb{W}_G from Example 2.12.

Definition 2.17. A *continuous coaction* of a quantum group $\mathbb{G} = (S, \Delta_S)$ on a C^* -algebra A is a homomorphism $\delta : A \rightarrow M(A \otimes S)$ such that

- (1) δ is 1-1
- (2) $(\text{id}_A \otimes \Delta_S)\delta = (\delta \otimes \text{id}_S)\delta$
- (3) $\delta(A) \cdot (1_A \otimes S) = A \otimes S$

We will sometimes refer to A as a \mathbb{G} - C^* -algebra.

We can also define coactions of quantum groups on C^* -correspondences:

Definition 2.18. A \mathbb{G} -equivariant C^* -correspondence over a \mathbb{G} - C^* -algebra (A, δ) is a C^* -correspondence X over A with a coaction $\sigma : X \rightarrow M(X \otimes S)$ such that

- (1) $\sigma(x)\delta(a) = \sigma(xa)$ and $\delta(a)\sigma(x) = \sigma(ax)$ for $x \in X$ and $a \in A$
- (2) $\delta(\langle x, y \rangle_A) = \langle \sigma(x), \sigma(y) \rangle_{M(A \otimes S)}$
- (3) $\sigma(x) \cdot (1 \otimes S) = X \otimes S$
- (4) $(1 \otimes S) \cdot \sigma(x) = X \otimes S$
- (5) $(\sigma \otimes \text{id}_A)\sigma = (\text{id}_X \otimes \sigma)\sigma$

2.4. Twisted Tensor Products. Throughout this section, $\mathbb{G} = (S, \Delta_S)$ and $\mathbb{H} = (T, \Delta_T)$ will be quantum groups, χ will be a bicharacter from $\mathbb{G} \rightarrow \widehat{\mathbb{H}}$, and A and B will be C^* -algebras carrying coactions δ_A and δ_B of \mathbb{G} and \mathbb{H} respectively.

Definition 2.19 (Definition 3.1 of [MRW14]). A χ -Heisenberg pair (or simply Heisenberg pair) is a pair of representations $\pi : S \rightarrow \mathcal{B}(\mathcal{H})$ and $\rho : T \rightarrow \mathcal{B}(\mathcal{H})$ such that

$$W_{1\pi}^S W_{2\rho}^T = W_{2\rho}^T W_{1\pi}^S \chi_{12} \in \mathcal{U}(\widehat{S} \otimes \widehat{T} \otimes \mathcal{K}(\mathcal{H}))$$

where $W_{1\pi}^S = ((\text{id}_{\widehat{S}} \otimes \pi)W^S)_{13}$ and $W_{2\rho}^T = ((\text{id}_{\widehat{T}} \otimes \rho)W^T)_{23}$.

Definition 2.20. Suppose (π, ρ) is a Heisenberg pair. Define maps

$$\begin{aligned} i_A : A &\rightarrow M(A \otimes B \otimes \mathcal{K}(\mathcal{H})) \\ i_B : B &\rightarrow M(A \otimes B \otimes \mathcal{K}(\mathcal{H})) \end{aligned}$$

as follows:

$$\begin{aligned} i_A(a) &= (\text{id}_A \otimes \pi)\delta_A(a)_{13} \\ i_B(b) &= (\text{id}_B \otimes \rho)\delta_B(b)_{23} \end{aligned}$$

It is shown in Lemma 3.20 of [MRW14] that $A \boxtimes_{\chi} B := i_A(A) \cdot i_B(B)$ is a C^* -subalgebra of $A \otimes B \otimes \mathbb{K}(\mathcal{H})$ and that, up to isomorphism, $A \boxtimes_{\chi} B$ does not depend upon the choice of Heisenberg pair. We refer to $A \boxtimes_{\chi} B$ as the *twisted tensor product* of A and B and we write $a \boxtimes b$ for $i_A(a)i_B(b)$.

This construction can also be extended to correspondences. Recall that, given a C^* -correspondence X over a C^* -algebra A , the *linking algebra* $L(X)$ is the algebra $\mathbb{K}(X \oplus A)$. This is often thought of in terms of its block matrix form: $\begin{bmatrix} \mathcal{K}(X) & X \\ \overline{X} & A \end{bmatrix}$. We will make use of the following proposition:

Proposition 2.21 (Proposition 2.7 of [BS89]). *Let $\delta_{L(X)} : L(X) \rightarrow M(L(X) \otimes S)$ be a coaction of \mathbb{G} on $L(X)$ such that the inclusion $j_A : A \rightarrow L(X)$ is \mathbb{G} -equivariant. Then there exists a unique coaction σ on X such that $j_{M(X \otimes S)} \circ \sigma = \delta_{L(X)} \circ j_X$ where $j_{M(X \otimes S)}$ is the inclusion $M(X \otimes S) \rightarrow M(L(X) \otimes S)$ and j_X is the inclusion $X \rightarrow L(X)$.*

Conversely, if σ is a coaction of \mathbb{G} on X , then there is a unique coaction $\delta_{L(X)}$ of \mathbb{G} on $L(X)$ such that $j_{M(X \otimes S)} \circ \sigma = \delta_{L(X)} \circ j_X$ and such that $j_A : A \rightarrow L(X)$ is \mathbb{G} -equivariant.

We refer the reader to [BS89] for the proof.

Definition 2.22. Let (X, A) and (Y, B) be C^* -correspondences with coactions of \mathbb{G} and \mathbb{H} respectively and let $\delta_{L(X)}$ and $\delta_{L(Y)}$ be the induced coactions of $L(X)$ and $L(Y)$. We can form the twisted tensor product $L(X) \boxtimes_{\chi} L(Y)$. Viewing X and Y as subspaces of $L(X)$ and $L(Y)$, we define

$$X \boxtimes_{\chi} Y := \iota_{L(X)}(X) \cdot \iota_{L(Y)}(Y)$$

Proposition 5.10 of [MRW14], and the discussion which follows it, shows that this is a correspondence over $A \boxtimes_{\chi} B$ with left action given by $\phi_X \boxtimes \phi_Y$. It also shows that

$$(22) \quad \mathcal{K}(X \boxtimes_{\chi} Y) \cong \mathcal{K}(X) \boxtimes_{\chi} \mathcal{K}(Y)$$

a fact which we shall make use of later.

3. DISCRETE GROUP TWISTED TENSOR PRODUCTS

3.1. Basics. In what follows we will restrict our attention to the following special case of the twisted tensor product construction:

Definition 3.1. Suppose that G is a discrete group, (A, G, α) is a C^* -dynamical system, and (B, G, δ) is a coaction. Let $\delta^{\alpha} : A \rightarrow M(A \otimes c_0(G))$ be the coaction of $c_0(G)$ (as a quantum group) on A associated to α as in Theorem 9.2.4 of [Tim08]. We can view δ as a coaction of the quantum group $C_r^*(G)$ and we can form the twisted tensor product $A \boxtimes_{W_G} B$ where W_G is the multiplicative unitary of Example 2.12 viewed as a bicharacter from $c_0(G)$ to $C_r^*(G)$ (in other words the reduced bicharacter of the quantum group $C_r^*(G)$). We refer to this special case as a *discrete group twisted tensor product*. Since this construction depends only upon the action and coaction, we will sometimes write $A_{\alpha} \boxtimes_{\delta} B$ for $A \boxtimes_{W_G} B$. We can also define a C^* -algebra by $B_{\delta} \boxtimes_{\alpha} A = B \boxtimes_{\widehat{W}_G} A$. It is easy to see that the map $a \boxtimes b \mapsto b \boxtimes a$ extends to an isomorphism $A_{\alpha} \boxtimes_{\delta} B \cong B_{\delta} \boxtimes_{\alpha} A$.

The main reason that this special case is of interest is that we can write down a precise formula for the multiplication and involution of certain elementary tensors. To understand this, we must recall that the coaction δ of a discrete group G on B gives rise to a G -grading of B . That is, there exist subspaces $\{B_s\}_{s \in G}$ such that

- (1) $\overline{\text{span}}(B_s) = B$
- (2) $B_s \cdot B_t \subseteq B_{st}$
- (3) $B_s^* = B_{s^{-1}}$.

Specifically, $B_s = \{b \in B : \delta(b) = b \otimes u_s\}$. With this in mind, we present the following:

Proposition 3.2. *Given a C^* -dynamical system (A, G, α) and a coaction (B, G, δ) , let $a, a' \in A$, $b_s \in B_s$ and $b \in B$. Then, in the twisted tensor product $A \boxtimes_{\delta} B$ we have:*

$$\begin{aligned} (a \boxtimes b_s)(a' \boxtimes b) &= a\alpha_s(a') \boxtimes b_s b \\ (a \boxtimes b_s)^* &= \alpha_{s^{-1}}(a)^* \boxtimes b_s^* \end{aligned}$$

Before we prove this, we will need the following:

Lemma 3.3. *Let G be a locally compact group and let $m : C_0(G) \rightarrow \mathbb{B}(L^2(G))$ be the left action of $C_0(G)$ on $L^2(G)$ by multiplication of functions in $L^2(G)$. Let $\lambda : C^*(G) \rightarrow \mathbb{B}(L^2(G))$ be the left regular representation. Then (m, λ) is a W_G -Heisenberg pair where W_G is the reduced bicharacter of G as in Example 2.15*

Proof. This is a special case of Example 3.9 of [MRW14]. \square

Lemma 3.4. *In the situation of the above proposition, let*

$$\begin{aligned} i_A : A &\rightarrow M(A \otimes B \otimes \mathbb{K}(L^2(G))) \\ i_B : B &\rightarrow M(A \otimes B \otimes \mathbb{K}(L^2(G))) \end{aligned}$$

be the maps associated to the Heisenberg pair (m, λ) as described in Definition 2.20. Then for any $a \in A$, $s \in G$ and $b_s \in B_s$ we have

$$i_B(b_s)i_A(a) = i_A(\alpha_{s^{-1}}(a))i_B(b_s)$$

Proof. Recall that (m, λ) is a covariant homomorphism $(c_0(G), G, \sigma) \rightarrow \mathbb{B}(L^2(G))$ where σ is left translation in $c_0(G)$. Also, $(\delta^\alpha, \sigma_2)$ is a covariant homomorphism $(A, G, \alpha) \rightarrow M(A \otimes c_0(G))$. Thus $((\text{id}_A \otimes m) \circ \delta^\alpha, \lambda_2)$ is a covariant homomorphism $(A, G, \alpha) \rightarrow M(A \otimes \mathbb{B}(L^2(G)))$. Thus

$$\lambda_2(s)^* \left((\text{id}_A \otimes m) \circ \delta^\alpha(a) \right) \lambda_2(s) = (\text{id}_A \otimes m) \circ \delta^\alpha(\alpha_s(a))$$

or equivalently

$$\left((\text{id}_A \otimes m) \circ \delta^\alpha(\alpha_{s^{-1}}(a)) \right) \lambda_2(s) = \lambda_2(s) \left((\text{id}_A \otimes m) \circ \delta^\alpha(a) \right)$$

With this in mind, we notice the following:

$$\begin{aligned} i_B(b_s)i_A(a) &= (\text{id}_B \otimes \lambda)\delta(b_s)_{23}(\text{id}_A \otimes m)\delta^\alpha(a)_{13} \\ &= (1_A \otimes b_s \otimes \lambda(s))(\text{id}_A \otimes m)\delta^\alpha(a)_{13} \\ &= (b_s)_2(\lambda(s))_3(\text{id}_A \otimes m)\delta^\alpha(a)_{13} \\ &= (b_s)_2(\text{id}_A \otimes m)\delta^\alpha(\alpha_{s^{-1}}(a))_{13}(\lambda(s))_3 \\ &= (\text{id}_A \otimes m)\delta^\alpha(\alpha_{s^{-1}}(a))_{13}(b_s)_2(\lambda(s))_3 \\ &= i_A(\alpha_{s^{-1}}(a))i_B(b_s) \end{aligned}$$

□

We can now prove Proposition 3.2.

Proof. (of Proposition 3.2)

We have:

$$\begin{aligned} (a \boxtimes b_s)(a' \boxtimes b) &= i_A(a)i_B(b_s)i_A(a')i_B(b) \\ &= i_A(a)i_A(\alpha_s(a'))i_B(b_s)i_B(b) \\ &= a\alpha_s(a') \boxtimes b_s b \end{aligned}$$

and

$$\begin{aligned} (a \boxtimes b_s)^* &= (i_A(a)i_B(b_s))^* \\ &= i_B(b_s^*)i_A(a^*) \\ &= i_A(\alpha_{s^{-1}}(a)^*)i_B(b_s^*) \\ &= \alpha_{s^{-1}}(a)^* \boxtimes b_s^* \end{aligned}$$

□

We also have simple formulas for the algebraic properties of twisted tensor products of correspondences:

Proposition 3.5. *Let (X, A) be a correspondence with an action (γ, α) of G and (Y, B) be a correspondence with a coaction (σ, δ) of G . Let $\alpha_{L(X)}$ be the action of G on $L(X)$ induced by the action of G on X and let $\delta_{L(Y)}$ be the coaction of G on $L(Y)$ induced by (σ, δ) . We can form the correspondence $X \gamma \boxtimes_{\sigma} Y := X \boxtimes_{W_G} Y \subseteq L(X) \boxtimes_{W_G} L(Y) = L(X) \alpha_{L(X)} \boxtimes_{\delta_{L(Y)}} L(Y)$ as in Definition 2.22. Then*

- (1) $(a \boxtimes b_s)(x \boxtimes y) = a\gamma_s(x) \boxtimes b_s y$
- (2) $(x \boxtimes y_s)(a \boxtimes b) = x\alpha_s(a) \boxtimes y_s b$
- (3) $\langle x \boxtimes y_s, x' \boxtimes y \rangle_{A \boxtimes B} = \alpha_{s^{-1}}(\langle x, x' \rangle_A) \boxtimes \langle y_s, y \rangle_B$

Proof. All of these facts follow from translating to multiplication in $L(X) \boxtimes L(Y)$ and applying Lemma 3.4. □

Corollary 3.6. *Let (X, A) be a correspondence with an action (γ, α) of G and (Y, B) be a correspondence with a coaction (σ, δ) of G . Suppose (X^0, A^0) is a generating system for (X, A) which is stable with respect to the group action. That is, $\gamma_s(X^0) \subseteq X^0$ and $\alpha_s(A^0) \subseteq A^0$ for all $s \in G$. Suppose further that (Y^0, B^0) is a generating system for (Y, B) such that the elements of Y^0 and B^0 are homogenous with respect to the grading. That is, for all $y \in Y^0$ there is $s \in G$ such that $y \in Y_s$*

and for all $b \in B^0$ there is $t \in G$ such that $b \in B_t$. Then

$$\begin{aligned}(X \boxtimes Y)_0 &:= \{x \boxtimes y : x \in X^0, y \in Y^0\} \\ (A \boxtimes B)_0 &:= \{a \boxtimes b : a \in A^0, b \in B^0\}\end{aligned}$$

form a generating system for $X \boxtimes Y$.

Proof. It is clear from the bilinearity of the twisted tensor product that $\overline{\text{span}}((X \boxtimes Y)_0) = X \boxtimes Y$ and $\overline{\text{span}}((A \boxtimes B)_0) = A \boxtimes B$. To see that $(X \boxtimes Y)_0$ is stable under the left and right actions of elements of $(A \boxtimes B)_0$, let $x \boxtimes y \in (X \boxtimes Y)_0$ and let $a \boxtimes b \in (A \boxtimes B)_0$. By definition, we must have that $b \in B_s$ and $y \in Y_t$ for some $s, t \in G$. Thus we have:

$$\begin{aligned}(a \boxtimes b)(x \boxtimes y) &= a\gamma_s(x) \boxtimes by \\ (x \boxtimes y)(a \boxtimes b) &= x\alpha_t(a) \boxtimes yb\end{aligned}$$

Since X^0 and A^0 are stable under the action of G , $\gamma_s(x) \in X^0$ and $\alpha_t(a) \in A^0$. Thus $a\gamma_s(x), \gamma_t(a)x \in X^0$ so $(X \boxtimes Y)_0$ is indeed stable under the left and right actions of $(A \boxtimes B)_0$. \square

3.2. Examples. In [MRW14], the authors show that if A and B are \mathbb{Z}_2 -graded algebras, then the graded tensor product $A \hat{\otimes} B$ is isomorphic $A \boxtimes_{W^{\mathbb{Z}_2}} B$ (where the coactions on A and B are the ones canonically associated with the grading) with the map $a \hat{\otimes} b \mapsto a \boxtimes b$ extending to an isomorphism. Since \mathbb{Z}_2 is self-dual, the coaction of \mathbb{Z}_2 on A gives rise to an action α of \mathbb{Z}_2 on A . If we let δ denote the coaction of \mathbb{Z}_2 on B we see that $A \boxtimes_{W^{\mathbb{Z}_2}} B = A \boxtimes_{\alpha, \delta} B$ so $A \hat{\otimes} B$ fits into our discrete group twisted tensor framework. The following example shows that the graded external tensor product of graded correspondences also fits into our framework.

Example 3.7. Let A and B be \mathbb{Z}_2 -graded C^* algebras and let X and Y be \mathbb{Z}_2 -graded correspondences over A and B (i.e. X and Y are graded as Hilbert A - and B -modules respectively and the left action maps ϕ_X and ϕ_Y are graded with respect to the induced gradings on $\mathcal{L}(X)$ and $\mathcal{L}(Y)$). Let $X^0 = X_0 \cup X_1$, $A^0 = A_0 \cup A_1$, $Y^0 = Y_0 \cup Y_1$ and $B^0 = B_0 \cup B_1$. Then by Corollary 2.10 (X^0, A^0) and (Y^0, B^0) are generating sets for X and Y respectively. Consider the graded external tensor product $X \hat{\otimes} Y$. This is the closure of the algebraic tensor product $X \odot Y$ with respect to the norm associated to the inner product whose value on generators is given by:

$$\langle x_1 \hat{\otimes} y_1, x_2 \hat{\otimes} y_2 \rangle = (-1)^{\partial y_1(\partial x_1 + \partial x_2)} \langle x_1, x_2 \rangle \hat{\otimes} \langle y_1, y_2 \rangle$$

where $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$. The left and right actions are given by:

$$\begin{aligned}(a \widehat{\otimes} b)(x \widehat{\otimes} y) &= (-1)^{\partial b \partial x} (ax \widehat{\otimes} by) \\ (x \widehat{\otimes} y)(a \widehat{\otimes} b) &= (-1)^{\partial y \partial a} (xa \widehat{\otimes} yb)\end{aligned}$$

for $a \in A_0$, $b \in B_0$, $x \in X_0$, and $y \in Y_0$. Thus

$$\begin{aligned}(A \widehat{\otimes} B)_0 &:= \{a \widehat{\otimes} b : a \in A_0, b \in B_0\} \\ (X \widehat{\otimes} Y)_0 &:= \{x \widehat{\otimes} y : x \in X_0, y \in Y_0\},\end{aligned}$$

is a generating system for $X \widehat{\otimes} Y$. Now, let (γ, α) be the action of \mathbb{Z}_2 on (X, A) associated to the grading of X and let (σ, δ) be the coaction of \mathbb{Z}_2 on (Y, B) associated to the grading of Y . Consider the associated twisted tensor product $X \boxtimes Y$. Note that the sets X_0 and A_0 are stable under the actions γ and α and that Y_0 and B_0 consist of elements which are homogeneous with respect to the gradings associated to σ and δ . Thus, by Corollary 3.6 the sets

$$\begin{aligned}(X \boxtimes Y)_0 &:= \{x \boxtimes y : x \in X_0, y \in Y_0\} \\ (A \boxtimes B)_0 &:= \{a \boxtimes b : a \in A_0, b \in B_0\}\end{aligned}$$

form a generating system for $X \boxtimes Y$. Let $\Phi_0 : (X \widehat{\otimes} Y)_0 \rightarrow (X \boxtimes Y)_0$ be the map $x \widehat{\otimes} y \mapsto x \boxtimes y$. This is clearly a bijection. Let $\varphi : A \widehat{\otimes} B \rightarrow A \boxtimes B$ be the isomorphism described above. For $a \in A_0$, $b \in B_0$, $x \in X_0$, and $y \in Y_0$, we have:

$$\begin{aligned}\Phi_0((a \widehat{\otimes} b)(x \widehat{\otimes} y)) &= (-1)^{\partial b \partial x} \Phi_0(ax \widehat{\otimes} by) \\ &= (-1)^{\partial b \partial x} (ax \boxtimes by) \\ &= a\gamma_{\partial b}(x) \boxtimes by \\ &= (a \boxtimes b)(x \boxtimes y) \\ &= \varphi(a \widehat{\otimes} b)\Phi_0(x \widehat{\otimes} y)\end{aligned}$$

and similarly

$$\Phi_0((x \widehat{\otimes} y)(a \widehat{\otimes} b)) = \Phi_0(x \widehat{\otimes} y)\varphi(a \widehat{\otimes} b)$$

therefore (Φ_0, φ) preserves the left and right actions. Additionally, for $x_1, x_2 \in X_0$ and $y_1, y_2 \in Y_0$, we have that:

$$\begin{aligned}
\langle \Phi_0(x_1 \hat{\otimes} y_1), \Phi_0(x_2 \hat{\otimes} y_2) \rangle &= \langle x_1 \boxtimes y_1, x_2 \boxtimes y_2 \rangle \\
&= \alpha_{\partial y_1}(\langle x_1, x_2 \rangle) \boxtimes \langle y_1, y_2 \rangle \\
&= (-1)^{\partial y_1(\partial x_1 + \partial x_2)} (\langle x_1, x_2 \rangle \boxtimes \langle y_1, y_2 \rangle) \\
&= (-1)^{\partial y_1(\partial x_1 + \partial x_2)} \varphi(\langle x_1, x_2 \rangle \hat{\otimes} \langle y_1, y_2 \rangle) \\
&= \varphi(\langle x_1 \hat{\otimes} x_2, y_1 \hat{\otimes} y_2 \rangle)
\end{aligned}$$

therefore, by Lemma 2.2, Φ_0 extends to an isomorphism $\Phi : X \hat{\otimes} Y \rightarrow X \boxtimes Y$. Thus $X \hat{\otimes} Y \cong X \boxtimes Y$.

The following example can be viewed as a generalization of the skew graph construction presented in Chapter 6 of [Rae05].

Example 3.8. Let $E = \{E^0, E^1, r_E, s_E\}$ and $F = \{F^0, F^1, r_F, s_F\}$ be directed graphs and let G be a discrete group. Let $A := c_0(E^0)$ and let $B := c_0(F^0)$. Let α^E be an action of G on E by graph automorphisms and let δ be a G -labeling of F , i.e. a map $\delta : F^1 \rightarrow G$. It is easy to see that the maps $f \mapsto f \circ \alpha_s^E$ on A together with the maps $x \mapsto x \circ \alpha_s^G$ on $c_c(E^1)$ give rise to group homomorphisms $G \rightarrow \text{Aut}(A)$ and $G \rightarrow \text{Aut}(c_c(E^1))$ which in turn give rise to a group action (γ, α) on $(X(E), A)$. We also get a coaction σ of G on $X(F)$ from δ . To see this, recall that $c_c(F^1)$ is generated by the characteristic functions χ_e and define $\sigma(\chi_f) := \chi_f \otimes u_{\delta(f)}$. So (σ, ι) is a coaction on $(X(F), B)$ where ι is the trivial coaction on B . We define a new directed graph $E_{\alpha^E \times \delta} F := \{E^0 \times F^0, E^1 \times F^1, r, s\}$ where $s(e \times f) = \alpha_{\delta(f)}^E(s_E(e)) \times s_F(f)$ and $r(e \times f) = r_E(e) \times r_F(f)$. We define $C := c_0(E^0 \times F^0) \cong A \otimes B$. We will show that $X(E_{\alpha^E \times \delta} F) \cong X(E) \gamma \boxtimes_\sigma X(F)$.

Let $X(E)^0$ be the set of characteristic functions on E^1 and let A^0 be the set of characteristic functions on E^0 . The characteristic functions densely span $c_c(E^1)$ which is dense in $X(E)$. For $\chi_v \in A^0$ and $\chi_e \in X(E)^0$ we have that

$$\begin{aligned}
\chi_v \cdot \chi_e &= \begin{cases} \chi_e & \text{if } r(e) = v \\ 0 & \text{otherwise} \end{cases} \in X(E)^0 \\
\chi_e \cdot \chi_v &= \begin{cases} \chi_e & \text{if } s(e) = v \\ 0 & \text{otherwise} \end{cases} \in X(E)^0 \\
\langle \chi_e, \chi_{e'} \rangle_A &= \begin{cases} \chi_{s(e)} & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

therefore $(X(E)^0, A^0)$ is a generating system for $X(E)$. We define $(X(F)^0, B^0)$ and $(X(E_{\alpha_E \times_\delta F})^0, C^0)$ in an analogous way, and we see that they are generating systems for $X(F)$ and $X(E_{\alpha_E \times_\delta F})$ respectively. Further, from the definition of the coaction σ we have that all characteristic functions on F^1 are homogeneous with respect to the grading induced by σ and since ι is trivial, the grading it induces is also trivial so A^0 is trivially homogeneous. Also, the actions α and γ take generating functions to generating functions: $\alpha_s : \chi_v \rightarrow \chi_{\alpha_{s^{-1}}(v)}$, $\gamma_s : \chi_e \rightarrow \chi_{\alpha_{s^{-1}}(e)}$ so the sets $X(E)^0$ and A^0 are fixed by the actions. This allows us to apply Corollary 3.6 and deduce that the sets

$$\begin{aligned} (X(E) \boxtimes X(F))^0 &= \{\chi_e \boxtimes \chi_f : e \in E^1, f \in F^1\} \\ (A \otimes B)^0 &= \{\chi_v \otimes \chi_w : v \in E^0, w \in F^0\} \end{aligned}$$

form a generating system for the twisted tensor product $X(E) {}_\gamma \boxtimes_\sigma X(F)$.

Let $\varphi : C \rightarrow A \otimes B$ and let $\Phi_0 : X(E_{\alpha_E \times_\delta F})^0 \rightarrow (X(E) \boxtimes X(F))^0$ be the map $\chi_{e \times f} \mapsto \chi_e \boxtimes \chi_f$. Clearly Φ_0 is bijective, we wish to show that it preserves the inner product and left and right actions. Note that

$$\begin{aligned} \varphi(\langle \chi_{e \times f}, \chi_{e' \times f'} \rangle_C) &= \begin{cases} \varphi(\chi_{s(e \times f)}) & \text{if } e \times f = e' \times f' \\ \phi(0) & \text{otherwise} \end{cases} \\ &= \begin{cases} \chi_{\alpha_{\delta(f)}^E(s_E(e))} \otimes \chi_{s_F(f)} & \text{if } e = e' \text{ and } f = f' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \alpha_{\delta(f)^{-1}}(\chi_{s_E(e)}) \otimes \chi_{s_F(f)} & \text{if } e = e' \text{ and } f = f' \\ 0 & \text{otherwise} \end{cases} \\ &= \alpha_{\delta(f)^{-1}}(\langle \chi_e, \chi_{e'} \rangle_A) \boxtimes \langle \chi_f, \chi_{f'} \rangle_B \\ &= \langle \chi_e \boxtimes \chi_f, \chi_{e'} \boxtimes \chi_{f'} \rangle_{A \otimes B} \\ &= \langle \Phi_0(\chi_{e \times f}), \Phi_0(\chi_{e' \times f'}) \rangle_{A \otimes B} \end{aligned}$$

and

$$\begin{aligned}
\Phi_0(\chi_{v \times w} \cdot \chi_{e \times f}) &= \begin{cases} \Phi_0(\chi_{e \times f}) & \text{if } r(e \times f) = v \times w \\ \Phi_0(0) & \text{otherwise} \end{cases} \\
&= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } r_E(e) = v \text{ and } r_F(f) = w \\ 0 & \text{otherwise} \end{cases} \\
&= (\chi_v \cdot \chi_e) \boxtimes (\chi_w \cdot \chi_f) \\
&= (\chi_v \otimes \chi_w)(\chi_e \boxtimes \chi_f) \\
&= \varphi(\chi_{v \times w})\Phi_0(\chi_{e \times f})
\end{aligned}$$

and finally

$$\begin{aligned}
\Phi_0(\chi_{e \times f} \cdot \chi_{v \times w}) &= \begin{cases} \Phi_0(\chi_{e \times f}) & \text{if } s(e \times f) = v \times w \\ \Phi_0(0) & \text{otherwise} \end{cases} \\
&= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } \alpha_{\delta(f)}^E(s_E(e)) = v \text{ and } s_F(f) = w \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \chi_e \boxtimes \chi_f & \text{if } s_E(e) = \alpha_{\delta(f)^{-1}}^E(v) \text{ and } s_F(f) = w \\ 0 & \text{otherwise} \end{cases} \\
&= \chi_e \cdot \chi_{\alpha_{\delta(f)^{-1}}^E(v)} \boxtimes \chi_f \cdot \chi_w \\
&= \chi_e \cdot \alpha_{\delta(f)}(\chi_v) \boxtimes \chi_f \cdot \chi_w \\
&= (\chi_e \boxtimes \chi_f)(\chi_v \boxtimes \chi_w) \\
&= \Phi_0(\chi_{e \times f})\varphi(\chi_{v \times w})
\end{aligned}$$

Therefore, by Lemma 2.2 we have that Φ_0 extends to a correspondence isomorphism $\Phi : X(E \times_{\alpha^E \times \delta} F) \rightarrow X(E) \gamma \boxtimes_{\sigma} X(F)$.

Example 3.9. In this example we will show that the crossed product of a correspondence by an action of a discrete group can be viewed as a discrete group twisted tensor product of correspondences. Suppose (γ, α) is an action of a discrete group G on a C^* -correspondence (X, A) . We wish to show that the reduced crossed product $X \rtimes_{\gamma, r} G$ is isomorphic to the twisted tensor product $X \gamma \boxtimes_{\delta_G} C_r^*(G)$ where we view $C_r^*(G)$ as a correspondence over itself.

Recall from Corollary 2.5 that the sets

$$\begin{aligned}
(X \rtimes_{\gamma, r} G)_0 &:= \{i_X(x)i_G^X(s) : x \in X, s \in G\} \\
(A \rtimes_{\alpha, r} G)_0 &:= \{i_A(a)i_G^A(s) : a \in A, s \in G\}
\end{aligned}$$

form a generating system for $X \rtimes_{\gamma, r} G$. Let $C_r^*(G)_0 = \{u_s : s \in G\}$, i.e. the image of G in $C_r^*(G)$. This set is closed under multiplication,

thus we may regard $(C_r^*(G)_0, C_r^*(G)_0)$ as a generating system for the correspondence $C_r^*(G)$. Furthermore, every element of $C_r^*(G)_0$ is homogeneous with respect to the grading arising from $\delta_G : u_s \mapsto u_s \otimes u_s$. Also, we may view (X, A) as a generating system for itself and then by Corollary 3.6 we see that the sets

$$\begin{aligned} (X \rtimes_{\gamma \boxtimes \delta_G} C_r^*(G))_0 &:= \{x \boxtimes u_s : x \in X, s \in G\} \\ (A \rtimes_{\alpha \boxtimes \delta_G} C_r^*(G))_0 &:= \{a \boxtimes u_s : a \in A, s \in G\} \end{aligned}$$

form a generating system for $X \rtimes_{\gamma \boxtimes \delta_G} C_r^*(G)$. We let φ be the isomorphism $A \rtimes_{\alpha, r} G \rightarrow A \rtimes_{\alpha \boxtimes \delta_G} C_r^*(G)$ and define $\Phi_0 : (X \rtimes_{\gamma, r} G)_0 \rightarrow (X \rtimes_{\gamma \boxtimes \delta_G} C_r^*(G))_0$ to be the map $i_X(x)i_G^X(s) \mapsto x \boxtimes u_s$. This is clearly a bijection, but we need to establish that it preserves the inner product and left and right actions. First, note that

$$\begin{aligned} \langle \Phi_0(i_X(x)i_G^X(s)), \Phi_0(i_X(y)i_G^X(t)) \rangle &= \langle x \boxtimes u_s, y \boxtimes u_t \rangle \\ &= \alpha_{s^{-1}}(\langle x, y \rangle) \boxtimes \langle u_s, u_t \rangle \\ &= \alpha_{s^{-1}}(\langle x, y \rangle) \boxtimes u_{s^{-1}t} \\ &= \varphi\left(i_A(\alpha_{s^{-1}}(\langle x, y \rangle))i_G^A(s^{-1}t)\right) \\ &= \varphi\left(\langle i_X(x)i_G^X(s), i_X(y)i_G^X(t) \rangle\right) \end{aligned}$$

also,

$$\begin{aligned} \Phi_0\left((i_X(x)i_G^X(s))(i_A(a)i_G^A(t))\right) &= \Phi_0(i_X(x\alpha_s(a))i_G^X(st)) \\ &= x\alpha_s(a) \boxtimes u_{st} \\ &= x\alpha_s(a) \boxtimes u_s u_t \\ &= (x \boxtimes u_s)(a \boxtimes u_t) \\ &= \Phi_0(i_X(x)i_G^X(s))\varphi(i_A(a)i_G^A(t)) \end{aligned}$$

and finally,

$$\begin{aligned} \Phi_0\left((i_A(a)i_G^A(s))(i_X(x)i_G^X(t))\right) &= \Phi_0(i_X(a\gamma_s(x))i_G^X(st)) \\ &= a\gamma_s(x) \boxtimes u_{st} \\ &= a\gamma_s(x) \boxtimes u_s u_t \\ &= (a \boxtimes u_s)(x \boxtimes u_t) \\ &= \varphi(i_A(a)i_G^A(s))\Phi_0(i_X(x)i_G^X(t)) \end{aligned}$$

Therefore, by Lemma 2.2, we have that Φ_0 extends to a correspondence isomorphism $\Phi : X \rtimes_{\gamma, r} G \rightarrow X \rtimes_{\gamma \boxtimes \delta_G} C_r^*(G)$.

Example 3.10. In this example we will see that crossed products by coactions on correspondences can also be viewed as twisted tensor products. Specifically, if (σ, δ) is a coaction of a discrete group G on a correspondence (X, A) , then we wish to show that $X \rtimes_{\sigma} G$ is isomorphic to $X \rtimes_{\sigma} \boxtimes_{\lambda} c_0(G)$ where we are viewing $c_0(G)$ as a correspondence over itself. To see this, recall from Corollary 2.10 that the sets

$$\begin{aligned} (X \rtimes_{\sigma} G)_0 &:= \{j_X(x_s)j_G^X(f) : x_s \in X_s, f \in c_0(G)\} \\ (A \rtimes_{\delta} G)_0 &:= \{j_A(a_s)j_G^A(f) : a_s \in A_s, f \in c_0(G)\} \end{aligned}$$

form a generating system for $X \rtimes_{\sigma} G$. Let $X^0 := \bigcup_{s \in G} X_s$ and let $A^0 := \bigcup_{s \in G} A_s$. The properties of the grading tell us that X^0 and A^0 densely span X and A and that $ax, xa \in X^0$ whenever $a \in A^0$ and $x \in X^0$. Thus (X^0, A^0) is a generating system for X which by definition consists of elements which are homogeneous with respect to the gradings of X and A . Viewing $(c_0(G), c_0(G))$ as a generating system for the correspondence $c_0(G)$, we may apply Corollary 3.6 and deduce that the sets

$$\begin{aligned} (X \rtimes_{\sigma} \boxtimes_{\lambda} c_0(G))_0 &:= \{x_s \boxtimes f : x_s \in X^0, f \in c_0(G)\} \\ (A \rtimes_{\delta} \boxtimes_{\lambda} c_0(G))_0 &:= \{a_s \boxtimes f : a_s \in A^0, f \in c_0(G)\} \end{aligned}$$

form a generating system for the twisted tensor product $X \rtimes_{\sigma} \boxtimes_{\lambda} c_0(G)$. Let φ be the isomorphism $A \rtimes_{\delta} G \rightarrow A \rtimes_{\delta} \boxtimes_{\lambda} c_0(G)$. We define $\Phi_0 : (X \rtimes_{\sigma} G)_0 \rightarrow (X \rtimes_{\sigma} \boxtimes_{\lambda} c_0(G))_0$ to be the map $j_X(x_s)j_G^X(f) \mapsto x_s \boxtimes f$. This is clearly bijective, but we must show that it preserves the inner product and left and right actions. To see this, suppose $x_s \in X_s \subseteq X^0$ and $x_t \in X_t \subseteq X^0$ and note that

$$\begin{aligned} \langle \Phi_0(j_X(x_s)j_G^X(f)), \Phi_0(j_X(x_t)j_G^X(g)) \rangle &= \langle x_s \boxtimes f, x_t \boxtimes g \rangle \\ &= \langle x_s, x_t \rangle \boxtimes \lambda_{s^{-1}}(\langle f, g \rangle) \\ &= \langle x_s, x_t \rangle \boxtimes \lambda_{s^{-1}}(\overline{f}g) \\ &= \varphi(j_A(\langle x_s, x_t \rangle)j_A^G(\lambda_{s^{-1}}(\overline{f}g))) \\ &= \varphi(\langle j_X(x_s)j_G^X(f), j_X(x_t)j_G^X(g) \rangle) \end{aligned}$$

furthermore, if $a_t \in A_t \subseteq A^0$ we have that

$$\begin{aligned} \Phi_0(j_X(x_s)j_G^X(f)(j_X(a_t)j_G^X(g))) &= \Phi_0(j_X(x_s a_t)j_G^X(\lambda_t(f)g)) \\ &= x_s a_t \boxtimes \lambda_t(f)g \\ &= (x_s \boxtimes f)(x_t \boxtimes g) \\ &= \Phi_0(j_X(x_s)j_G^X(f))\varphi(j_A(a_t)j_G^A(g)) \end{aligned}$$

and

$$\begin{aligned}
\Phi_0\left((j_A(a_t)j_G^A(f))(j_X(x_s)j_G^X(g))\right) &= \Phi_0\left(j_X(a_tx_s)j_G^X(\lambda_s(f)g)\right) \\
&= a_tx_s \boxtimes \lambda_s(f)g \\
&= (a_t \boxtimes f)(x_s \boxtimes g) \\
&= \varphi(j_A(a_t)j_G^A(f))\Phi_0(j_X(x_s)j_G^X(g))
\end{aligned}$$

Therefore, by Lemma 2.2, we have that Φ_0 extends to a correspondence isomorphism $\Phi : X \rtimes_\sigma G \rightarrow X \boxtimes_\lambda c_0(G)$.

4. BALANCED TWISTED TENSOR PRODUCTS

Throughout this section, G will be a discrete group, Z will be a compact abelian group, (A, G, α) , (A, Z, μ) and (B, Z, ν) will be dynamical systems and (B, G, δ) will be a coaction such that μ commutes with α and ν is covariant with respect to δ .

Proposition 4.1. $(A \boxtimes_\delta B, Z, \lambda)$ is a dynamical system where

$$\lambda_z = \mu_z \boxtimes \nu_{z^{-1}}$$

Proof. Since μ commutes with α and ν is covariant with respect to δ , we know that each map λ_z is an automorphism. To show that $z \mapsto \lambda_z$ is a group homomorphism, note that

$$\begin{aligned}
\lambda_z \circ \lambda_w(a \boxtimes b) &= \lambda_z(\mu_w(a) \boxtimes \nu_{w^{-1}}(b)) \\
&= \mu_z \circ \mu_w(a) \boxtimes \nu_{z^{-1}} \circ \nu_{w^{-1}}(b) \\
&= \mu_{zw}(a) \boxtimes \nu_{z^{-1}w^{-1}}(b) \\
&= \mu_{zw}(a) \boxtimes \nu_{w^{-1}z^{-1}}(b) \\
&= \mu_{zw}(a) \boxtimes \nu_{(zw)^{-1}}(b) \\
&= \lambda_{zw}(a \boxtimes b)
\end{aligned}$$

□

Definition 4.2. We call the fixed point algebra $(A \boxtimes B)^\lambda$ of the above action the Z -balanced twisted tensor product of A and B and we denote it by $A \boxtimes_Z B$.

Proposition 4.3. Let $a \boxtimes b \in A \boxtimes_Z B$. Then $\mu_z(a) \boxtimes b, a \boxtimes \nu_z(b) \in A \boxtimes_Z B$ for all $z \in Z$ and, in fact, $\mu_z(a) \boxtimes b = a \boxtimes \nu_z(b)$ for all $z \in Z$.

Proof. Let $w \in Z$. Note that

$$\begin{aligned}
 \lambda_w(\mu_z(a) \boxtimes b) &= \mu_w \circ \mu_z(a) \boxtimes \nu_{w^{-1}}(b) \\
 &= \mu_z \circ \mu_w(a) \boxtimes \nu_{w^{-1}}(b) \\
 &= (\mu_z \boxtimes \text{id}_B)(\lambda_w(a \boxtimes b)) \\
 &= (\mu_z \boxtimes \text{id}_B)(a \boxtimes b) \\
 &= \mu_z(a) \boxtimes b
 \end{aligned}$$

Thus $\mu_z(a) \boxtimes b$ is fixed under the action of λ and is therefore an element of $A \boxtimes_Z B$. Showing that $a \boxtimes \nu_z(b) \in A \boxtimes_Z B$ is similar. To show that these are actually equivalent, notice that

$$\begin{aligned}
 \mu_z(a) \boxtimes b &= \lambda_{z^{-1}}(\mu_z(a) \boxtimes b) \\
 &= \mu_{z^{-1}} \circ \mu_z(a) \boxtimes \nu_z(b) \\
 &= a \boxtimes \nu_z(b)
 \end{aligned}$$

□

Proposition 4.4. *Keeping the above conventions, let δ^μ and δ^ν be the dual coactions. Since Z is compact, \widehat{Z} will be discrete and thus δ^μ and δ^ν give gradings $\{A_\chi\}_{\chi \in \widehat{Z}}$ and $\{B_\chi\}_{\chi \in \widehat{Z}}$ of A and B . For each $\chi \in \widehat{Z}$ let*

$$S_\chi := \{a \boxtimes b : a \in A_\chi, b \in B_\chi\}$$

and let $S := \bigcup_{\chi \in \widehat{Z}} S_\chi$. Then $A \boxtimes_Z B = \overline{\text{span}}(S)$.

Proof. First we will show that $\overline{\text{span}}(S) \subseteq A \boxtimes_Z B$. Let $a \boxtimes b \in S$. Then $a \boxtimes b \in S_\chi$ for some $\chi \in \widehat{Z}$. Thus for all $z \in Z$ we have that

$$\begin{aligned}
 \lambda_z(a \boxtimes b) &= \mu_z(a) \boxtimes \nu_{z^{-1}}(b) \\
 &= \chi(z)a \boxtimes \chi(z^{-1})b \\
 &= \chi(z)\chi(z^{-1})(a \boxtimes b) \\
 &= a \boxtimes b
 \end{aligned}$$

thus $S \subseteq A \boxtimes_Z B$ and therefore $\overline{\text{span}}(S) \subseteq A \boxtimes_Z B$.

To show the reverse inclusion, let $c \in A \boxtimes_Z B$. Then $c \approx \sum a_i \boxtimes b_i$. Since the subspaces $\{A_\chi\}_{\chi \in \widehat{Z}}$ span A and the subspaces $\{B_\chi\}_{\chi \in \widehat{Z}}$ span B , we may assume without loss of generality that there are $\chi_i, \chi'_i \in \widehat{Z}$ such that $a_i \in A_{\chi_i}$ and $b_i \in B_{\chi'_i}$ for each i . Let $\Upsilon : A \boxtimes B \rightarrow A \boxtimes_{\mathbb{T}} B$ be the conditional expectation $d \mapsto \int_Z \lambda_z(d) dz$. Since c is assumed to be in $A \boxtimes_{\mathbb{T}} B$, we have that $c = \Upsilon(c)$. Thus, using the continuity and

linearity of Υ we have the following

$$\begin{aligned}
c &= \Upsilon(c) \\
&= \int_Z \lambda_z(c) dz \\
&\approx \int_Z \lambda_z \left(\sum_i a_{\chi_i} \boxtimes b_{\chi'_i} \right) dz \\
&= \sum_i \int_Z \lambda_z(a_{\chi_i} \boxtimes b_{\chi'_i}) dz \\
&= \sum_i \int_Z \mu_z(a_{\chi_i}) \boxtimes \nu_{z^{-1}}(b_{\chi'_i}) dz \\
&= \sum_i \int_Z (\chi_i(z) a_{\chi_i}) \boxtimes (\chi'_i(z^{-1}) b_{\chi'_i}) dz \\
&= \sum_i (a_{\chi_i} \boxtimes b_{\chi'_i}) \int_Z \chi_i(z) \chi'_i(z^{-1}) dz
\end{aligned}$$

But the integral $\int_Z \chi_i(z) \chi'_i(z^{-1}) dz$ is equal to 1 if $\chi_i = \chi'_i$ and zero otherwise (this is a consequence of the Peter-Weyl theorem). Thus c can be approximated by a sum of elements from S :

$$c \approx \sum_i a_{\chi_i} \boxtimes b_{\chi_i}$$

therefore $A \boxtimes_{\mathbb{T}} B \subseteq \overline{\text{span}}(S)$ so $A \boxtimes_{\mathbb{T}} B = \overline{\text{span}}(S)$. \square

Proposition 4.5. *Let $\{S_\chi\}_{\chi \in \widehat{Z}}$ be as above. This defines a coaction of \widehat{Z} on $A \boxtimes_Z B$ and thus also an action γ of Z on $A \boxtimes_Z B$. We have that $\gamma = \mu \boxtimes \text{id}_B = \text{id}_A \boxtimes \nu$.*

Proof. Let $c \in A \boxtimes_{\mathbb{T}} B$. By the previous proposition we can approximate $c \approx \sum_i a_{\chi_i} \boxtimes b_{\chi_i}$. Then, by continuity and linearity

$$\begin{aligned}
\gamma_z(c) &\approx \sum_i \gamma_z(a_{\chi_i} \boxtimes b_{\chi_i}) \\
&= \sum_i \chi_i(z) (a_{\chi_i} \boxtimes b_{\chi_i}) \\
&= \sum_i (\chi_i(z) a_{\chi_i}) \boxtimes b_{\chi_i} \\
&= \sum_i \mu_z(a_{\chi_i}) \boxtimes b_{\chi_i}
\end{aligned}$$

Thus $\gamma = \mu \boxtimes \text{id}_B$. Showing that $\gamma = \text{id}_A \boxtimes \nu$ is similar. \square

Lemma 4.6. *Suppose $\{A_\chi\}_{\chi \in \widehat{Z}}$ and $\{B_\chi\}_{\chi \in \widehat{Z}}$ are saturated gradings of A and B , that is $A_\chi A_\omega = A_{\chi\omega}$. Then $\{S_\chi\}_{\chi \in \widehat{Z}}$ (as described above) is saturated.*

Proof. We already have that $S_\chi S_\omega \subseteq S_{\chi\omega}$ for all $\chi, \omega \in \widehat{Z}$ so it will suffice to show the reverse inclusion. Let $a \boxtimes b \in S_{\chi\omega}$. Then $a \in A_{\chi\omega}$ and $b \in B_{\chi\omega}$. Since the gradings of A and B are saturated, we have that $a \approx \sum_i a_{\chi,i} a_{\omega,i}$ and $b \approx \sum_j b_{\chi,j} b_{\omega,j}$ with $a_{\chi,i} \in A_\chi$, $a_{\omega,i} \in A_\omega$, $b_{\chi,j} \in B_\chi$ and $b_{\omega,j} \in B_\omega$. Thus

$$\begin{aligned} a \boxtimes b &= \sum_{i,j} a_{\chi,i} a_{\omega,i} \boxtimes b_{\chi,j} b_{\omega,j} \\ &= \sum_{i,j} (a_{\chi,i} \boxtimes b_{\chi,j}) (a_{\omega,i} \boxtimes b_{\omega,j}) \\ &\subseteq S_\chi \boxtimes S_\omega \end{aligned}$$

\square

5. IDEAL COMPATIBILITY

Before we state our main result, we need to define two technical conditions involving the Katsura ideals. We call these conditions Katsura nondegeneracy and ideal compatibility. These conditions are basically the same as those presented in the author's first paper [Mor15], only the definition of ideal compatibility must be modified to allow twisted tensor products instead of ordinary ones.

Definition 5.1. Let (X, A) be a correspondence and let J_X be the Katsura ideal of A (recall that the Katsura ideal is the ideal $J_X = \phi^{-1}(\mathcal{K}(X)) \cap (\ker(\phi))^\perp$). We say that X is *Katsura nondegenerate* if $J_X \cdot X = X$.

Example 5.2. Let X be a correspondence over a C^* -algebra A such that the left action is injective and implemented by compacts. In this case we have that $J_X = A$. Thus:

$$\begin{aligned} X \cdot J_X &= X \cdot A \\ &= X \end{aligned}$$

Definition 5.3. Recall that a vertex in a directed graph is called a *source* if it receives no edges. We will call such a vertex a *proper source* if it emits at least one edge.

Proposition 5.4. *Let E be a directed graph. Then $X(E)$ is Katsura nondegenerate if and only if E has no proper sources and no infinite receiver emits an edge.*

Proof. Suppose there is $v \in E^0$ such that $|r^{-1}(v)| = \infty$ and $|s^{-1}(v)| > 0$. Then for every $f \in J_X$ we have $f(v) = 0$. Thus for any $g \in C_c(E^1)$, $f \in J_X$, and $e \in s^{-1}(v)$, we have $(g \cdot f)(e) = g(e)f(s(e)) = g(e)f(v) = 0$. Thus $h(e) = 0$ for all $h \in C_c(E^1) \cdot J_X$ and, taking the limit, $x(e) = 0$ for all $x \in X \cdot J_X$. Thus $\delta_e \notin X \cdot J_X$ since $\delta_e(e) = 1 \neq 0$ but $\delta_e \in X$. Therefore $X \neq X \cdot J_X$, i.e. X is not Katsura nondegenerate.

Similarly, suppose that E has a proper source v . Then, since $|r^{-1}(v)| = 0$ we must have $f(v) = 0$ for all $f \in J_X$. Then for any $g \in C_c(E^1)$ and $e \in s^{-1}(v)$ we have that $(g \cdot f)(e) = g(e)f(v) = 0$ for $f \in J_X$. Thus by similar reasoning as above we have that $x(e) = 0$ for all $x \in X \cdot J_X$ and so $\delta_e \notin X \cdot J_X$ but $\delta_e \in X$ and we can again conclude that $X \neq X \cdot J_X$ so X is not Katsura nondegenerate.

On the other hand, suppose E has no proper sources and no infinite receiver in E emits an edge. Let $e \in E^1$ and let $v = s(e)$. Then $|r^{-1}(v)| < \infty$ and $|r^{-1}(v)| > 0$ by assumption, so function in J_X can be supported on v . In particular, $\delta_v \in J_X$. Since $\delta_e \cdot \delta_v = \delta_e$ we know that $\delta_e \in X \cdot J_X$. Since e was arbitrary, we have that all such characteristic functions are contained in $X \cdot J_X$. But these functions densely span $C_c(E^1)$ and thus densely span X , so we have that $X \subseteq X \cdot J_X$ and therefore $X = X \cdot J_X$ so X is Katsura nondegenerate. \square

We now turn our attention to ideal compatibility:

Definition 5.5. Given two correspondences (X, A) and (Y, B) , an action (γ, α) of a discrete group G on X and a coaction (σ, δ) of G on Y , we say that X and Y are *ideal compatible* with respect to γ and σ if $J_X \alpha_\delta J_Y = J_{X \gamma_\sigma Y}$ where J_X, J_Y and $J_{X \gamma_\sigma Y}$ are the Katsura ideals of X, Y , and $X \gamma_\sigma Y$ respectively.

Once again, this condition will be met in the case of injective left actions implemented by compacts:

Proposition 5.6. *Suppose we have correspondences (X, A) and (Y, B) , an action (γ, α) of a discrete group G on X and a coaction (σ, δ) of G on Y . If the left actions of A on X and B on Y are injective and implemented by compacts, then X and Y are ideal compatible.*

Proof. In this case the Katsura ideals are the whole algebras: $J_X = A$, $J_Y = B$ and $J_{X \gamma_\sigma Y} = A \alpha_\delta B$. Thus the equation $J_X \alpha_\delta J_Y = J_{X \gamma_\sigma Y}$ becomes trivial. \square

In Example 8.13 of [Rae05], it is shown that if E is a discrete graph, then

$$J_{X(E)} = \overline{\text{span}}\{\chi_v : 0 < |r^{-1}(v)| < \infty\}$$

where $X(E)$ is the associated correspondence and $\chi_v \in c_0(E^0)$ denotes the characteristic function of the vertex $v \in E^0$. With this in mind, we give the following proposition:

Proposition 5.7. *Let E and F be discrete graphs and let $(X = X(E), A = c_0(E^0))$ and $(Y = X(F), B = c_0(F^0))$ be the associated correspondences. Suppose α^E is an action of a discrete group G on E and let (γ, α) be the associated action on (X, A) . Let $\delta : F^1 \rightarrow G$ be a labeling of the edges of F and let (σ, ι) be the associated coaction on (Y, B) . Then X and Y are ideal compatible with respect to γ and σ .*

Proof. Recall that $X \rtimes_{\gamma} Y = X(E \rtimes_{\alpha^E \times \delta} F)$ where $E \rtimes_{\alpha^E \times \delta} F$ is the graph defined in Example 3.8. Thus

$$J_{X \rtimes_{\gamma} Y} = \overline{\text{span}}\{\chi_{v \times w} : 0 < |r_{E \rtimes_{\alpha^E \times \delta} F}^{-1}(v \times w)| < \infty\}$$

By definition, $r_{E \rtimes_{\alpha^E \times \delta} F} = r_E \times r_F$ so $r_{E \rtimes_{\alpha^E \times \delta} F}^{-1}(v \times w) = r_E^{-1}(v) \times r_F^{-1}(w)$ and thus $|r_{E \rtimes_{\alpha^E \times \delta} F}^{-1}(v \times w)| = |r_E^{-1}(v)| \cdot |r_F^{-1}(w)|$. But $0 < |r_E^{-1}(v)| \cdot |r_F^{-1}(w)| < \infty$ if and only if $0 < |r_E^{-1}(v)| < \infty$ and $0 < |r_F^{-1}(w)| < \infty$. Thus we have that

$$J_{X \rtimes_{\gamma} Y} = \overline{\text{span}}\{\chi_{v \times w} : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\}$$

If we identify $c_0(E^0 \times F^0)$ with $c_0(E^0) \otimes c_0(F^0)$ in the standard way, we see that $\chi_{v \times w} = \chi_v \otimes \chi_w$ (recall from Example 3.8 that ι is the trivial coaction so we have that $A \rtimes_{\alpha} B \cong A \otimes B$). Thus

$$\begin{aligned} J_{X \rtimes_{\gamma} Y} &= \overline{\text{span}}\{\chi_v \otimes \chi_w : 0 < |r_E^{-1}(v)|, |r_F^{-1}(w)| < \infty\} \\ &= \overline{\text{span}}\{f \otimes g : f \in J_X, g \in J_Y\} \\ &= J_X \otimes J_Y \end{aligned}$$

Therefore, X and Y are ideal-compatible. \square

6. MAIN RESULT

We will now state our main theorem, although we delay the proof until later in this section. Throughout this section X and Y will be correspondences over C^* -algebras A and B , (σ, δ) will be a coaction of a discrete group G on Y and (γ, α) will be an action of G on X . We will denote the induced action on \mathcal{O}_X by γ' and the induced coaction on \mathcal{O}_Y by σ' . To simplify the notation, we will make the following definitions: $X \boxtimes Y := X \rtimes_{\gamma} Y$, $A \boxtimes B := A \rtimes_{\alpha} B$, and $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y := \mathcal{O}_{X \rtimes_{\gamma} Y} \rtimes_{\sigma', \mathbb{T}} \mathcal{O}_Y$.

Theorem 6.1. *Suppose X and Y are full, ideal compatible, and Katsura nondegenerate (see the previous section). Then $\mathcal{O}_{X \boxtimes_\sigma Y} \cong \mathcal{O}_{X \boxtimes_{\gamma', \mathbb{T}} \sigma', \mathbb{T}} \mathcal{O}_Y$.*

Before we can prove this result, we will need the following lemmas:

Lemma 6.2. *Suppose (π_X, ψ_X) and (π_Y, ψ_Y) are Toeplitz representations of X and Y in C^* -algebras C and D and let γ' and σ' be the induced action and coaction on C and D . Let $\psi := \psi_X \otimes \psi_Y$ and let $\pi := \pi_X \otimes \pi_Y$. Then (π, ψ) is a Toeplitz representation of $X \boxtimes_\sigma Y$ in $C \boxtimes_{\gamma', \sigma'} D$.*

Proof. Let $y \in Y_s$ for some $s \in G$,

$$\begin{aligned} \psi((x \boxtimes y)(a \boxtimes b)) &= \psi(x\alpha_s(a) \boxtimes yb) \\ &= \psi_X(x\alpha_s(a)) \boxtimes \psi_Y(yb) \\ &= \psi_X(x)\pi_X(\alpha_s(a)) \boxtimes \psi_Y(y)\pi_Y(b) \\ &= \psi_X(x)\gamma'_s(\pi_X(a)) \boxtimes \psi_Y(y)\pi_Y(b) \\ &= (\psi_X(x) \boxtimes \psi_Y(y))(\pi_X(a) \boxtimes \pi_Y(b)) \\ &= \psi(x \boxtimes y)\pi(a \boxtimes b) \end{aligned}$$

A similar argument with $b \in B_s$ shows that

$$\psi((a \boxtimes b)(x \boxtimes y)) = \pi(a \boxtimes b)\psi(x \boxtimes y)$$

Further,

$$\begin{aligned} \psi(x \boxtimes y)^* \psi(x' \boxtimes y') &= (\psi_X(x) \boxtimes \psi_Y(y))^* (\psi_X(x') \boxtimes \psi_Y(y')) \\ &= (\gamma'_{s-1}(\psi_X(x)^*) \boxtimes \psi_Y(y)^*) (\psi_X(x') \boxtimes \psi_Y(y')) \\ &= \gamma'_{s-1}(\psi_X(x)^*) \gamma'_{s-1}(\psi_X(x')) \boxtimes \psi_Y(y^*) \psi_Y(y') \\ &= \gamma'_{s-1}(\psi_X(x)^* \psi_X(x')) \boxtimes \psi_Y(y)^* \psi_Y(y') \\ &= \gamma'_{s-1}(\pi_X(\langle x, x' \rangle_A)) \boxtimes \pi_Y(\langle y, y' \rangle_B) \\ &= \pi_X(\gamma_{s-1}(\langle x, x' \rangle_A)) \boxtimes \pi_Y(\langle y, y' \rangle_B) \\ &= \pi(\gamma_{s-1}(\langle x, x' \rangle_A) \boxtimes \langle y, y' \rangle_B) \\ &= \pi(\langle x \boxtimes y, x' \boxtimes y' \rangle_{A \boxtimes B}) \end{aligned}$$

These equalities can be extended extend linearly and continuously (by the linearity and continuity of the maps $\psi_X, \pi_X, \psi_Y, \pi_Y, \psi$, and π) to all of $X \boxtimes Y$ and $A \boxtimes B$ thus (π, ψ) is a Toeplitz representation. \square

Lemma 6.3. *If (π, ψ) is the Toeplitz representation in Lemma 6.2, then:*

$$\psi^{(1)}(k(S \boxtimes T)) = \psi_X^{(1)}(S) \boxtimes \psi_Y^{(1)}(T)$$

for all $S \in \mathcal{K}(X)$ and $T \in \mathcal{K}(Y)$.

Proof. Without loss of generality we may assume that $S = \Theta_{x,x'}$ and $T = \Theta_{y,y'}$ with $y \in Y_s$ and $y' \in Y_t$. We have:

$$\begin{aligned} \psi^{(1)}(k(\Theta_{x,x'} \boxtimes \Theta_{y,y'})) &= \psi^{(1)}(\Theta_{x \boxtimes y, \gamma_{ts-1}(x') \boxtimes y'}) \\ &= \psi(x \boxtimes y) \psi(\gamma_{ts-1}(x') \boxtimes y')^* \\ &= (\psi_X(x) \boxtimes \psi_Y(y)) (\psi_X(\gamma_{ts-1}(x')) \boxtimes \psi_Y(y'))^* \\ &= (\psi_X(x) \boxtimes \psi_Y(y)) (\gamma'_{t-1}(\psi_X(\gamma_{ts-1}(x'))^*) \boxtimes \psi_Y(y')^*) \\ &= (\psi_X(x) \boxtimes \psi_Y(y)) (\gamma'_{s-1}(\psi_X(x')^*) \boxtimes \psi_Y(y')^*) \\ &= (\psi_X(x) \psi_X(x')^*) \boxtimes (\psi_Y(y) \psi_Y(y')^*) \\ &= \psi_X^{(1)}(\Theta_{x,x'}) \boxtimes \psi_Y^{(1)}(\Theta_{y,y'}) \end{aligned}$$

□

Lemma 6.4. *Suppose that X and Y are ideal compatible. If (π_X, ψ_X) and (π_Y, ψ_Y) are Cuntz-Pimsner covariant then so is (π, ψ) .*

Proof. Let ϕ_X , ϕ_Y , and ϕ be the left action maps on X , Y and $X \boxtimes_\sigma Y$ respectively. Note that from the definition of $X \boxtimes_\sigma Y$ we have that $\phi = j \circ (\phi_X \boxtimes \phi_Y)$ and thus, in particular $\phi|_{J_X \boxtimes Y} = k \circ (\phi_X|_{J_X} \boxtimes \phi_Y|_{J_Y})$. Let $c \in J_X \boxtimes Y$. Since $J_X \boxtimes Y = J_X \boxtimes J_Y$ we may approximate $c \approx \sum a_i \boxtimes b_i$ for some $a_i \in J_X$ and $b_i \in J_Y$. We have

$$\begin{aligned} \psi^{(1)}(\phi(c)) &\approx \sum \psi^{(1)}(\phi(a_i \boxtimes b_i)) \\ &= \sum \psi^{(1)}(k(\phi_X(a_i) \boxtimes \phi_Y(b_i))) \\ &= \sum \psi_X^{(1)}(\phi_X(a_i)) \boxtimes \psi_Y^{(1)}(\phi_Y(b_i)) \\ &= \sum \pi_X(a_i) \boxtimes \pi_Y(b_i) \\ &= \sum \pi(a_i \boxtimes b_i) \\ &\approx \pi(c) \end{aligned}$$

thus (π, ψ) is Cuntz-Pimsner covariant. □

We are now ready to prove our main theorem.

Proof. (of Theorem 6.1) First, we will construct a $*$ -homomorphism $F : \mathcal{O}_{X \boxtimes Y} \rightarrow \mathcal{O}_X \boxtimes \mathcal{O}_Y$. Let (k_X, k_A) and (k_Y, k_B) be the standard Cuntz-Pimsner covariant representations of X and Y in \mathcal{O}_X and \mathcal{O}_Y and let $\psi := k_X \boxtimes k_Y$ and $\pi := k_A \boxtimes k_B$. Then by Lemma 6.4, (ψ, π) is a Cuntz-Pimsner covariant representation of $X \boxtimes Y$ in $\mathcal{O}_X \boxtimes \mathcal{O}_Y$. Thus, by the universal property there is a unique homomorphism $F : \mathcal{O}_{X \boxtimes Y} \rightarrow \mathcal{O}_X \boxtimes \mathcal{O}_Y$ such that

$$(\psi, \pi) = (F \circ k_{X \boxtimes Y}, F \circ k_{A \boxtimes B})$$

The gauge actions Γ_X and Γ_Y on \mathcal{O}_X and \mathcal{O}_Y give rise to \mathbb{Z} gradings $\{\mathcal{O}_X^n\}_{n \in \mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n \in \mathbb{Z}}$. By Proposition 4.4 the sets

$$S_n := \{x \boxtimes y : x \in \mathcal{O}_X^n, y \in \mathcal{O}_Y^n\}$$

give a \mathbb{Z} -grading of $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Notice that we have the following:

$$(23) \quad F(k_{A \boxtimes B}(a \boxtimes b)) = \pi(a \boxtimes b) = (k_A \boxtimes k_B)(a \boxtimes b) \in S_0$$

$$(24) \quad F(k_{X \boxtimes Y}(x \boxtimes y)) = \psi(x \boxtimes y) = (k_X \boxtimes k_Y)(x \boxtimes y) \in S_1$$

For $a \in A, b \in B, x \in X, y \in Y$. Since the image of F is generated by $F(k_{A \boxtimes B}(A \boxtimes B))$ and $F(k_{X \boxtimes Y}(X \boxtimes Y))$ we see that the image of F lies inside $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Furthermore, note that the grading $\{S_n\}_{n \in \mathbb{Z}}$ gives rise to an action Γ of \mathbb{T} such that $\Gamma_z(s) = z^n s$ for $s \in S_n$. In particular, by (23) we have that $\Gamma_z(\pi(c)) = \pi(c)$ for all $c \in A \boxtimes B$ and by (24) we have that $\Gamma_z(\psi(w)) = z\psi(w)$ for all $w \in X \boxtimes Y$. Thus Γ is a gauge action for (ψ, π) . Since k_A, k_B, k_X , and k_Y are all injective, ψ and π will be injective. By the Gauge Invariant Uniqueness Theorem, we have that F is injective.

It remains to show that F is surjective onto $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$. Recall that, since X and Y are full, so are the gradings $\{\mathcal{O}_X^n\}_{n \in \mathbb{Z}}$ and $\{\mathcal{O}_Y^n\}_{n \in \mathbb{Z}}$ and thus by Lemma 4.6 so is the grading $\{S_n\}_{n \in \mathbb{Z}}$. Thus $\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_Y$ is generated by the elements of S_1 (see Lemma 3.7 of [Mor15]). Therefore, to show that F is surjective, it suffices to show that S_1 is in the image of F .

Recall that \mathcal{O}_X^1 is densely spanned by elements of the form: $k_X^{n+1}(x)k_X^n(x')^*$. Similarly, \mathcal{O}_Y^1 is densely spanned by elements of the form $k_Y^{n+1}(y)k_Y^n(y')^*$. Thus the elements

$$(25) \quad k_X^{n+1}(x)k_X^n(x')^* \boxtimes k_Y^{m+1}(y)k_Y^m(y')^*$$

are dense in S_1 . Fix such an element. We may assume without loss of generality that y is a tensor product of homogeneous elements (for if not, it can be approximated by a sum of such products). In other words, we may assume that $y = y_1 \otimes \cdots \otimes y_{m+1}$ where each y_i homogeneous

with respect to the G -grading of Y . Let us say that $y_i \in Y_{s_i}$ for each i . Let $s := s_1 \cdots s_{m+1}$. Then

$$k_Y^{m+1}(y) = k_Y(y_1) \cdots k_Y(y_{m+1}) \in \mathcal{O}_Y^s$$

Similarly, we can assume that $y' = y'_1 \otimes \cdots \otimes y'_m$ with $y'_i \in Y_{t_i}$ and then $k_Y^m(y') \in \mathcal{O}_Y^t$ where $t := t_1 \cdots t_m$.

Let us assume $n \leq m$. Then $m = n + l$ for some $l \geq 0$. Then we can factor $y = y^{(1)} \otimes y^{(2)}$ and $y' = y'^{(1)} \otimes y'^{(2)}$. Since X is Katsura nondegenerate, we can factor $x = x_0 a$ and $x' = x'_0 a'$ with $x_0, x'_0 \in X$ and $a, a' \in J_X$. With this in mind, we can factor (25) as follows:

$$\begin{aligned} & k_X^{n+1}(x) k_X^n(x')^* \boxtimes k_Y^{m+1}(y) k_Y^m(y')^* \\ &= (k_X^{n+1}(x) \boxtimes k_Y^{m+1}(y)) \left(\gamma'_{s^{-1}}(k_X^n(x'))^* \boxtimes k_Y^m(y')^* \right) \\ &= (k_X^{n+1}(x_0) k_A(a) \boxtimes k_Y^{n+1}(y^{(1)}) k_Y^l(y^{(2)})) \\ &\quad \left(\gamma'_{s^{-1}}(k_A(a'))^* \gamma'_{s^{-1}}(k_X^n(x'_0))^* \boxtimes k_Y^l(y'^{(2)})^* k_Y^n(y'^{(1)})^* \right) \\ &= \left(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)}) \right) \left(\gamma'_{s_1^{-1}}(k_A(a)) \boxtimes k_Y^l(y^{(2)}) \right) \\ &\quad \left(\gamma'_{s^{-1}}(k_A(a'))^* \boxtimes k_Y^l(y'^{(2)})^* \right) \left(\gamma'_{t_2 s^{-1}}(k_X^n(x'_0))^* \boxtimes k_Y^n(y'^{(1)})^* \right) \\ &= \left(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)}) \right) \left(\gamma'_{s_1^{-1}}(k_A(a)) \gamma'_{s_2 s^{-1}}(k_A(a'))^* \boxtimes k_Y^l(y^{(2)}) k_Y^l(y'^{(2)})^* \right) \\ &\quad \left(\gamma'_{t_2 s^{-1}}(k_X^n(x'_0))^* \boxtimes k_Y^n(y'^{(1)})^* \right) \\ &= \left(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)}) \right) \left(k_A(\alpha_{s_1^{-1}}(a) \alpha_{s_2 s^{-1}}(a')^*) \boxtimes k_Y^l(y^{(2)}) k_Y^l(y'^{(2)})^* \right) \\ &\quad \left(\gamma'_{t_1 t_2 s^{-1}}(k_X^n(x'_0)) \boxtimes k_Y^n(y'^{(1)}) \right)^* \\ &= \left(k_X^{n+1}(x_0) \boxtimes k_Y^{n+1}(y^{(1)}) \right) \left(k_X^{(1)} \circ \phi_X(\alpha_{s_1^{-1}}(a) \alpha_{s_2 s^{-1}}(a')^*) \boxtimes k_Y^{(1)}(\Theta_{y^{(2)}, y'^{(2)}}) \right) \\ &\quad \left(k_X^n(\alpha_{t s^{-1}}(x'_0)) \boxtimes k_Y^n(y'^{(1)}) \right)^* \\ &= \psi^{n+1}(x_0 \boxtimes y^{(1)}) \psi^{(1)} \left(\phi_X(\alpha_{s_1^{-1}}(a) \alpha_{s_2 s^{-1}}(a')^*) \boxtimes \Theta_{y^{(2)}, y'^{(2)}} \right) \\ &\quad \psi^n(\alpha_{t s^{-1}}(x'_0) \boxtimes y'^{(1)})^* \end{aligned}$$

Since all three factors in the final line are in the algebra generated by (ψ, π) , we know that $k_X^{n+1}(x) k_X^n(x')^* \boxtimes k_Y^{m+1}(y) k_Y^m(y')^*$ must be in the image of F . If $n > m$ we can factor $x = x^{(1)} \otimes x^{(2)}$, $x' = x'^{(1)} \otimes x'^{(2)}$, $y = y_0 a$, and $y' = y'_0 a'$ and apply a similar argument. So the image of F generates S_1 and thus F is surjective onto $\mathcal{O}_{X \times_{\gamma'} Y}$. Hence we have established that $\mathcal{O}_{X \times_{\gamma'} Y} \cong \mathcal{O}_X \times_{\gamma', \sigma'} \mathcal{O}_Y$ \square

7. EXAMPLES

In this section, we will apply our main theorem to some of the examples of twisted tensor products of correspondences we discussed earlier.

Example 7.1. Let X and Y be \mathbb{Z}_2 -graded correspondences over \mathbb{Z}_2 -graded C^* -algebras A and B . As in Example 3.7, let (γ, α) be the action of \mathbb{Z}_2 on (X, A) associated to the grading on X and let (σ, δ) be the coaction of \mathbb{Z}_2 on (Y, B) associated to the grading of Y . Let γ' be the action of \mathbb{Z}_2 on \mathcal{O}_X induced by (γ, α) and let σ' be the coaction of \mathbb{Z}_2 on \mathcal{O}_Y induced by (σ, δ) . Suppose X and Y are Katsura nondegenerate and ideal compatible with respect to σ and γ . Then by our main result together with Example 3.7, we see that $\mathcal{O}_{X \widehat{\otimes} Y} \cong \mathcal{O}_{X \boxtimes Y} \cong \mathcal{O}_X \boxtimes_{\gamma', \mathbb{T}} \mathcal{O}_Y$.

Note that γ' and σ' give rise to \mathbb{Z}_2 -gradings on \mathcal{O}_X and \mathcal{O}_Y so it makes sense to speak of the \mathbb{Z}_2 -graded tensor product $\mathcal{O}_X \widehat{\otimes} \mathcal{O}_Y$. Let $\varphi : \mathcal{O}_X \widehat{\otimes} \mathcal{O}_Y \rightarrow \mathcal{O}_X \boxtimes \mathcal{O}_Y$ be the map $w \widehat{\otimes} z \mapsto w \boxtimes z$. Note that

$$\begin{aligned} \varphi((w_1 \widehat{\otimes} z_1)(w_2 \widehat{\otimes} z_2)) &= (-1)^{\partial z_1 \partial w_2} \varphi(w_1 w_2 \widehat{\otimes} z_1 z_2) \\ &= (-1)^{\partial z_1 \partial w_2} (w_1 w_2 \boxtimes z_1 z_2) \\ &= w_1 \gamma'_{\partial z_1}(w_2) \boxtimes z_1 z_2 \\ &= (w_1 \boxtimes z_1)(w_2 \boxtimes z_2) \\ &= \varphi(w_1 \widehat{\otimes} z_1) \varphi(w_2 \widehat{\otimes} z_2) \end{aligned}$$

and also

$$\begin{aligned} \varphi((w \widehat{\otimes} z)^*) &= (-1)^{\partial w \partial z} \varphi(w^* \widehat{\otimes} z^*) \\ &= (-1)^{\partial w \partial z} (w^* \boxtimes z^*) \\ &= (-1)^{\partial w^* \partial z^*} (w^* \boxtimes z^*) \\ &= \gamma_{\partial z^*}(w^*) \boxtimes z^* \\ &= (w \boxtimes z)^* \\ &= \varphi(w \widehat{\otimes} z)^* \end{aligned}$$

therefore φ is a $*$ -homomorphism. Further, φ takes a densely spanning set (the elementary tensors $w \widehat{\otimes} z$) in $\mathcal{O}_X \widehat{\otimes} \mathcal{O}_Y$ bijectively onto a densely spanning set (the elementary tensors $w \boxtimes z$) in $\mathcal{O}_X \boxtimes_{\gamma', \mathbb{T}} \mathcal{O}_Y$ so φ is an isomorphism. Thus we have that

$$\mathcal{O}_{X \widehat{\otimes} Y} \cong \mathcal{O}_X \widehat{\otimes}_{\mathbb{T}} \mathcal{O}_Y$$

Example 7.2. In this example we will show that the graph algebra of the product graph constructed in Example 3.8 is a twisted tensor product of the graph algebras of the underlying graphs. Suppose E and F are

directed graphs, α^E an action of a discrete group G on E and let δ be a labelling of the edges of F by elements of G . Let (γ, α) and (σ, ι) be the associated action and coaction on the graph correspondences $(X, A) = (X(E), c_0(E^0))$ and $(Y, B) = (X(F), c_0(F^0))$. Suppose further, that E and F have no sinks, no proper sources (see Definition 5.3), and that no infinite receiver in E or F emits an edge. Then we have that the graph correspondences (X, A) and (Y, B) are full, and by Propositions 5.4 and 5.6 they are Katsura nondegenerate and ideal compatible with respect to γ and σ . Thus we may apply our main result. Let γ' be the induced action on $\mathcal{O}_X \cong C^*(E)$ and σ' be the induced coaction on $\mathcal{O}_Y \cong C^*(F)$. We have

$$C^*(E \times_{\alpha^E \times \delta} F) \cong \mathcal{O}_{X \boxtimes_{\sigma} Y} \cong \mathcal{O}_X \boxtimes_{\gamma', \mathbb{T}} \mathcal{O}_Y \cong C^*(E) \boxtimes_{\gamma', \mathbb{T}} C^*(F)$$

Before we see how our main result applies to crossed products, we will need a few lemmas.

Lemma 7.3. *Let (A, G, α) be a dynamical system, (B, G, δ) a coaction and C a C^* -algebra. There exists an isomorphism*

$$\sigma_{23} : A \boxtimes_{\Sigma_{23}(\delta \otimes \text{id}_C)} (B \otimes C) \rightarrow (A \otimes C) \boxtimes_{\alpha \otimes \iota} B$$

Proof. By definition,

$$A \boxtimes_{\Sigma_{23}(\delta \otimes \text{id}_C)} (B \otimes C) = i_A(A) \cdot i_{B \otimes C}(B \otimes C) \subseteq A \otimes B \otimes C \otimes \mathcal{B}(L^2(G))$$

where $i_A = \delta_{1m}^\alpha$ and

$$i_{B \otimes C} = \circ(\Sigma_{23}(\delta \otimes \text{id}_C))_{23\lambda}$$

. Similarly,

$$(A \otimes C) \boxtimes_{\alpha \otimes \iota} B = i_{A \otimes C}(A \otimes C) \cdot i_B(B) \subseteq A \otimes C \otimes B \otimes \mathcal{B}(L^2(G))$$

where $i_{A \otimes C} = m_4 \circ \delta_{12m}^{\alpha \otimes \iota}$ and $i_B = \lambda_4 \circ \delta_{3\lambda}$. Since $\Sigma_{23} : A \otimes B \otimes C \otimes \mathcal{B}(L^2(G)) \rightarrow A \otimes C \otimes B \otimes \mathcal{B}(L^2(G))$ is an isomorphism, we can let σ_{23} be the restriction of Σ_{23} to the subalgebra $A \boxtimes_{\Sigma_{23}(\delta \otimes \text{id}_C)} (B \otimes C)$. From there it suffices to show that the image of σ_{23} is equal to $(A \otimes C) \boxtimes_{\alpha \otimes \iota} B$. $A \boxtimes_{\Sigma_{23}(\delta \otimes \text{id}_C)} (B \otimes C)$ is densely spanned by elements of the form $i_A(a)i_{B \otimes C}(b \otimes c)$ and $(A \otimes C) \boxtimes_{\alpha \otimes \iota} B$ is densely spanned by elements of the form $i_{A \otimes C}(a \otimes c)i_B(b)$. Since σ_{23} is a restriction of the linear norm preserving map Σ_{23} , it too is linear and norm preserving, thus to show that the image of σ_{23} is $(A \otimes C) \boxtimes_{\alpha \otimes \iota} B$, it suffices to show that

$\sigma_{23}(i_A(a)i_{B \otimes C}(b \otimes c)) = i_{A \otimes C}(a \otimes c)i_B(b)$. To see this, note that

$$\begin{aligned}
\sigma_{23}(i_A(a)i_{B \otimes C}(b \otimes c)) &= \Sigma_{23} \left(\delta^\alpha(a)_{1m} (\Sigma_{23}(\delta \otimes \text{id}_C)(b \otimes c))_{23\lambda} \right) \\
&= \Sigma_{23} \left(\delta^\alpha(a)_{1m} (\Sigma_{23}(\delta(b) \otimes c))_{23\lambda} \right) \\
&= \delta^\alpha(a)_{1m} \Sigma_{23}(c_3 \delta(b)_{2\lambda}) \\
&= \delta^\alpha(a)_{1m} c_2 \delta(b)_{3\lambda} \\
&= \left(\Sigma_{23}(\delta^\alpha(a) \otimes c) \right)_{12m} \delta(b)_{3\lambda} \\
&= \delta^{\alpha \otimes \iota}(a \otimes c)_{12m} \delta(b)_{3\lambda} \\
&= i_{A \otimes C}(a \otimes c) i_B(b)
\end{aligned}$$

thus $\sigma_{23} : A \boxtimes_{\alpha \otimes \iota, \Sigma_{23}(\delta \otimes \text{id}_C)} (B \otimes C) \rightarrow (A \otimes C) \boxtimes_{\alpha \otimes \iota, \delta} B$ is an isomorphism. \square

Lemma 7.4. *Let (γ, α) be an action of a discrete group G on a correspondence (X, A) and let (B, G, δ) be a coaction. If $J_{X, \gamma \boxtimes_\delta B} = J_{X, \alpha \boxtimes_\delta B}$ and J_X is Katsura-nondegenerate, then*

$$\mathcal{O}_{X, \gamma \boxtimes_\delta B} \cong \mathcal{O}_{X, \gamma' \boxtimes_\delta B}$$

where B is viewed as a correspondence over itself in the left-hand-side and γ' is the induced action on \mathcal{O}_X .

Proof. In this context, our main result gives us

$$\mathcal{O}_{X, \gamma \boxtimes_\delta B} \cong \mathcal{O}_{X, \gamma \boxtimes_{\delta, \mathbb{T}}} \mathcal{O}_B$$

Recall that $\mathcal{O}_B \cong B \otimes C(\mathbb{T})$ and the gauge action corresponds to $\iota \otimes \lambda$ where ι is the trivial action and λ is left translation. Using the isomorphism between $C(\mathbb{T})$ and $C^*(\mathbb{Z})$, we see that

$$\begin{aligned}
\mathcal{O}_X \boxtimes_{\mathbb{T}} \mathcal{O}_B &\cong \mathcal{O}_X \boxtimes_{\mathbb{T}} (B \otimes C^*(\mathbb{Z})) \\
&= \overline{\text{span}} \{ \mathcal{O}_X^n \boxtimes (B \otimes C^*(\mathbb{Z})^n) \} \\
&= \overline{\text{span}} \{ x \boxtimes (b \otimes u_n) \in \mathcal{O}_X \boxtimes (B \otimes C^*(\mathbb{Z})) : x \in \mathcal{O}_X^n \}
\end{aligned}$$

where u_n is the unitary in $C^*(\mathbb{Z})$ associated to $n \in \mathbb{Z}$. Applying the isomorphism σ_{23} from the previous lemma, we see that the above is isomorphic to

$$\begin{aligned}
&\overline{\text{span}} \{ (x \otimes u_n) \boxtimes b \in (\mathcal{O}_X \otimes C^*(\mathbb{Z})) \boxtimes B : x \in \mathcal{O}_X^n \} \\
&= \overline{\text{span}} \{ x \otimes u_n \in \mathcal{O}_X \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n \} \boxtimes B
\end{aligned}$$

The grading $\{\mathcal{O}_X^n\}_{n \in \mathbb{Z}}$ corresponds to a coaction ε such that $\varepsilon(x) = x \otimes u_n$ for all $x \in \mathcal{O}_X^n$. Continuing from above, we have:

$$\begin{aligned} & \overline{\text{span}}\{x \otimes u_n \in \mathcal{O}_X \otimes C^*(\mathbb{Z}) : x \in \mathcal{O}_X^n\} \boxtimes B \\ &= \overline{\text{span}}\{\varepsilon(x) : x \in \mathcal{O}_X^n\} \boxtimes B \\ &= \varepsilon(\mathcal{O}_X) \boxtimes B \\ &\cong \mathcal{O}_X \boxtimes B \end{aligned}$$

□

Lemma 7.5. *Let (σ, δ) be a coaction of a discrete group G on a full correspondence (X, A) . Let α be an action of G on a C^* -algebra B . Suppose $J_{X \rtimes_{\sigma} B} = J_{X \rtimes_{\delta} B}$ and J_X is Katsura-nondegenerate. Then*

$$\mathcal{O}_{X \rtimes_{\sigma} B} \cong \mathcal{O}_{X \rtimes_{\sigma'} B}$$

where σ' is the induced coaction on \mathcal{O}_X

Proof. The proof is similar to the proof of Lemma 7.4. □

Example 7.6. Let (X, A) be a correspondence and let (γ, α) be an action of a discrete group G on A . Suppose further, that $J_{X \rtimes_{\gamma, r} G} = J_X \rtimes_{\alpha, r} G$. Recall that Example 3.9 showed that $X \rtimes_{\gamma, r} G \cong X \rtimes_{\gamma} C_r^*(G)$. Thus we have

$$\mathcal{O}_{X \rtimes_{\gamma, r} G} \cong \mathcal{O}_{X \rtimes_{\gamma} C_r^*(G)}$$

Since $A \rtimes_{\alpha, r} G \cong A \rtimes_{\alpha} C_r^*(G)$, the condition that $J_{X \rtimes_{\gamma, r} G} = J_X \rtimes_{\alpha, r} G$ can be restated as $J_{X \rtimes_{\gamma} C_r^*(G)} = J_X \rtimes_{\alpha} C_r^*(G)$. Thus the correspondences X and $C_r^*(G)$ to be ideal compatible. Applying Lemma 7.4, we have that $\mathcal{O}_{X \rtimes_{\gamma} C_r^*(G)} \cong \mathcal{O}_{X \rtimes_{\gamma'} C_r^*(G)}$ and this, in turn, is isomorphic to $\mathcal{O}_{X \rtimes_{\gamma', r} G}$. Thus we have:

$$\mathcal{O}_{X \rtimes_{\gamma, r} G} \cong \mathcal{O}_{X \rtimes_{\gamma', r} G}$$

Example 7.7. Suppose (σ, δ) is a coaction of a discrete group G on a correspondence (X, A) . Then $X \rtimes_{\sigma} G \cong X \rtimes_{\sigma} c_0(G)$ by Proposition 3.10. If $J_{X \rtimes_{\sigma} c_0(G)} = J_X \rtimes_{\delta} c_0(G)$ (or in other terms $J_{X \rtimes_{\sigma} G} = J_X \rtimes_{\delta} G$) then Lemma 7.5 tells us that

$$\mathcal{O}_{X \rtimes_{\sigma} c_0(G)} \cong \mathcal{O}_{X \rtimes_{\sigma'} c_0(G)}$$

which we can restate as

$$\mathcal{O}_{X \rtimes_{\sigma} G} \cong \mathcal{O}_{X \rtimes_{\sigma'} G}$$

where σ' is the induced coaction on \mathcal{O}_X .

These last two examples are not new. In [HN08] it was shown that $\mathcal{O}_{X \rtimes_{\gamma} G} \cong \mathcal{O}_X \rtimes_{\gamma'} G$ for any locally compact amenable group, and in [Bns] it is shown that $\mathcal{O}_{X \rtimes_{\gamma,r} G} \cong \mathcal{O}_X \rtimes_{\gamma',r} G$ for any locally compact group provided that $J_{X \rtimes_{\gamma,r} G} = J_X \rtimes_{\alpha,r} G$. In [KQR12] it was shown that $\mathcal{O}_{X \rtimes_{\sigma} G} \cong \mathcal{O}_X \rtimes_{\sigma'} G$ for full (not reduced) coactions σ provided certain technical conditions involving the Katsura ideal are satisfied. Even though our main result does not recover these results in their full generality, it seems reasonable to hope that the main result of this paper might be extended to arbitrary locally compact groups and perhaps even quantum groups. In this case we would have a single framework in which to describe all results of this type.

REFERENCES

- [Bns] Bédos, Erik and Kaliszewski, S and Quigg, John and Robertson, David, *A new look at crossed product correspondences and associated c -algebras*, Journal of Mathematical Analysis and Applications **426** (Journal of Mathematical Analysis and Applications), no. 2, 1080–1098.
- [BS89] Saad Baaĵ and Georges Skandalis, *C^* -algèbres de Hopf et théorie de Kasparov*, *K-Theory* **2** (1989), no. 6, 683–721.
- [EKQR02] S. Echterhoff, S. Kaliszewski, J. Quigg, and I. Raeburn, *A categorical approach to imprimitivity theorems for C^* -dynamical systems*, *Dynamical Systems* **180** (2002).
- [HN08] Gai Hao and Chi-Keung Ng, *Crossed products of C^* -correspondences by amenable*, *J. Math. Anal. Appl.* **345** (2008), no. 2, 702–707.
- [KQR12] S Kaliszewski, J Quigg, and D Robertson, *Coactions on cuntz-pimsner algebras*, arXiv preprint arXiv:1204.5822 (2012).
- [Kat03] T Katsura, *A construction of C^* -algebras from C^* -correspondences*, *Contemporary Mathematics* **335** (2003), 173–182.
- [Kat04] T. Katsura, *On C^* -algebras associated with C^* -correspondences*, *Journal of Functional Analysis* **217** (2004), no. 2, 366–401.
- [Lan95] E. C. Lance, *Hilbert C^* -modules a toolkit for operator algebraists*, *London Mathematical Society Lecture Note Series*, vol. 210, Cambridge University Press, Cambridge, 1995.
- [Mor15] Adam Morgan, *Cuntz-pimsner algebras associated to tensor products of correspondences*, arXiv preprint arXiv:1510.04959 (2015).
- [MRW14] Ralf Meyer, Sutanu Roy, and Stanisław Lech Woronowicz, *Quantum group-twisted tensor products of c^* -algebras*, *International Journal of Mathematics* (2014).
- [Qui96] J. C. Quigg, *Discrete C^* -coactions and C^* -algebraic bundles*, *Journal of the Australian Mathematical Society (Series A)* **60** (1996), no. 02, 204–221.
- [Rae05] I. Raeburn, *Graph algebras*, *Regional Conference Series in Mathematics*, no. 103, American Mathematical Society, 2005.
- [Tim08] Thomas Timmermann, *An invitation to quantum groups and duality*, *EMS Textbooks in Mathematics*, European Mathematical Society (EMS), Zürich, 2008.

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